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Cooperative filtering for parameter identification of diffusion processes

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Abstract— **This paper presents a cooperative filtering scheme for online parameter identification of 2D diffusion processes using data collected by a mobile sensor network moving in the diffusion field. The diffusion equation is incorporated into the information dynamics associated with the trajectories of the mobile sensors. A cooperative Kalman filter is developed to provide estimates of field values, the gradient, and the temporal variations of the field values along the trajectories. This leads to a co-design scheme for state estimation and parameter identification for diffusion processes that is different from using static sensors. Utilizing the state estimates from the filters, a recursive least square (RLS) algorithm is designed to estimate the unknown diffusion coefficient of the field. A set of sufficient conditions is derived for the convergence of the cooperative Kalman filter. Simulation results show satisfactory performance of the proposed method.**

I. INTRODUCTION

Many environmental processes are characterized by both spatial and temporal correlations and often represented mathematically by partial differential equations (PDEs). One of the typical PDEs is the diffusion equation, which has been widely used to model the diffusion phenomena such as the propagation of chemical contaminates in the water or in the air [1]. In many practical problems, some parameters in the diffusion equation such as the diffusion coefficient may be unknown or inaccurate, which requires identification or calibration [2]. For instance, in the application of smog modeling and prediction, the unknown diffusivity tensor in an advection-diffusion PDE can be estimated based on measurements collected by monitoring stations [3].

There exist many contributions on the issue of parameter identification of PDEs using static sensor networks [4]–[6]. However, in missions of modeling a relatively large region, the use of static sensor networks is often impractical due to the sheer size of the fields and high cost of installing enough static sensors to ensure coverage of the entire field [7] [8]. For parameter identification purpose, a preferable opportunity is using mobile sensor networks, which are collections of robotic agents with computational, communication, sensing, and locomotive capabilities [2]. There has been an immense interest in the use of mobile sensor networks to detect, monitor, and model the environment [2], [9]–[14]. One of the general approaches of parameter identification is to first decide optimal locations or trajectories of sensors offline, then, formulate a least square problem and search for the parameters that minimize the error between measurements of the true state (with true parameters) and the estimated state [2], [15]. This is usually referred to as performing the twin experiments in data assimilation literature [16]. To find parameters that minimize the least square cost function, PDEs have to be solved using finite element methods over the entire spatial domain, and the optimal solution is obtained by numerical methods for each time step, which requires high computational power. In many realistic scenarios, it is desirable and more practical to achieve parameter estimation while a mobile sensor network is exploring a field instead of estimating parameters offline. For example, in chemical plume tracking, the mobile sensor network has no prior knowledge of the diffusion process, thus, it's preferable that the mobile sensor network can estimate the unknown diffusion coefficient while detecting and tracking the plume to obtain real-time information about the process. Therefore, we aim to develop an online parameter identification algorithm that estimates parameters iteratively, which can save tremendous computational power.

In this paper, we develop a cooperative filtering scheme for online parameter identification of diffusion processes using a mobile sensor network. We first incorporate the diffusion equation into the information dynamics and develop a cooperative Kalman filter. Compared to the cooperative Kalman filter in [10], the proposed filter deals with a diffusion field instead of a static field and involves more states, which enables the online estimation of the temporal variations of the field values along the trajectory of a mobile sensor network. Meanwhile, we provide a method to estimate the Hessian that is necessary for the cooperative Kalman filter. Utilizing the estimates from the filters, we employ the recursive least square (RLS) method to iteratively update the estimate of the unknown parameter in the diffusion equation. We justify a set of sufficient conditions regarding the formation shape and motion of the mobile sensor network that guarantee the convergence of the proposed filter. Finally, we present simulation results to show the satisfactory performance.

The problem is formulated in Section II. Section III presents the cooperative filtering design and Section IV shows the convergence proof. Simulation results are presented in Section V and conclusions follow in Section VI.

II. PROBLEM FORMULATION

A. *Diffusion processes*

Consider the two-dimensional (2D) diffusion process defined on a domain $\Omega \in \mathbb{R}^2$:

$$\frac{\partial z(r,t)}{\partial t} = \theta \nabla^2 z(r,t),\ r \in \Omega, \tag{1}$$

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where $z(r,t)$ is the concentration function, $\theta > 0$ is a constant diffusion coefficient, and ∇^2 represents the Laplacian operator. In practical applications such as environmental monitoring, the exploration domain Ω is much larger than the source and sensor dimensions so that the boundary can be modeled as a flat surface. Thus, we assume Dirichlet boundary conditions on the boundary $\partial\Omega$ [17],

$$z(r,t) = 0, \; r \in \partial\Omega. \tag{2}$$

Many natural processes can be described by the diffusion equation (1). In many scenarios, θ is unknown or inaccurate, which requires identification or estimation.

B. Sensor dynamics

Consider a formation of N coordinated sensing agents moving in the field, each of which carries a sensor that takes point measurements of the field $z(r,t)$. We consider the sensing agents with single-integrator dynamics given by

$$\dot{r}_i(t) = u_i(t), i = 1,2,...,N, \tag{3}$$

where $r_i(t)$ and $u_i(t) \subseteq \mathbb{R}^2$ are the position and the velocity of the ith agent, respectively.

In most applications, the sensor measurements are taken discretely over time. Let the moment when new measurements are available be t_k, where k is an integer index. Denote the position of the ith agent at time t_k be r_i^k and the field value at r_i^k be $z(r_i^k,k)$. Let $r_c^k = [r_{c,x}^k, r_{c,y}^k]^T$ be the center of the formation at time t_k, i.e., $r_c^k = \frac{1}{N}\sum_{i=1}^N r_i^k$. The measurement of the ith agent can be modeled as

$$p(r_i^k,k) = z(r_i^k,k) + n_i, \tag{4}$$

where n_i is assumed to be i.i.d. Gaussian noise. We have the following assumption for the sensing agents.

Assumption II.1 *Each agent can obtain its position r_i^k and the measurement of concentration value $z(r_i^k,k)$, and share these information with other agents.*

The problem is formulated as: under Assumption II.1, develop an online parameter identification algorithm that estimates the unknown constant diffusion coefficient θ of the diffusion equation (1) based on the information collected by a mobile sensor network moving in the diffusion field.

III. Cooperative filtering for parameter identification

This section describes our approach for parameter identification of the diffusion model (1). Before discussing the algorithm, we first introduce the numerical scheme for the diffusion equation. We then derive the information dynamics and construct a cooperative Kalman filter. Next, we show how to estimate the Hessian that is necessary for the cooperative Kalman filter. Finally, we show how to use the RLS method to iteratively update the estimate of θ.

A. Numerical scheme for the diffusion equation

In order to run RLS to estimate θ based on discrete measurements, we discretize the diffusion PDE (1) at the formation center r_c^k. The discretization of the PDE requires approximating spatial and temporal derivatives by means of finite difference method.

Assume the current time step is k. Then the term $\frac{\partial z(r_c,t)}{\partial t}$ can be approximated by

$$\frac{\partial z(r_c,t)}{\partial t}\Big|_{t=t_k} \approx \frac{z(r_c^k,k+1) - z(r_c^k,k)}{t_s}, \tag{5}$$

and $\nabla^2 z(r_c,t)$ can be expressed with a finite difference approximation as:

$$\begin{aligned}\nabla^2 z(r_c,t) &\approx \frac{\partial^2 z(r_c,t)}{\partial^2 r_{c,x}} + \frac{\partial^2 z(r_c,t)}{\partial^2 r_{c,y}} \\ &\approx \frac{z(r_c^k - \Delta r_x \cdot x,k) - 2z(r_c^k,k) + z(r_c^k + \Delta r_x \cdot x,k)}{\Delta r_x^2} \\ &+ \frac{z(r_c^k - \Delta r_y \cdot y,k) - 2z(r_c^k,k) + z(r_c^k + \Delta r_y \cdot y,k)}{\Delta r_y^2},\end{aligned} \tag{6}$$

where t_s is the sampling period, $x = [1 \;\, 0]^T$ and $y = [0 \;\, 1]^T$ are unit axis vectors, Δr_x and Δr_y are the grid sizes along the x and y directions, $z(r_c^k,k)$ is the field value at position r_c^k at the current time step k, and $z(r_c^k,k-1)$ is the field value at position r_c^k at the previous time step $k-1$. By employing formation control, we can arrange four agents in a symmetric formation as shown in Fig. 1. Then we have $\Delta r_x = \|r_c^k - r_2^k\| = \|r_c^k - r_4^k\|$, $\Delta r_y = \|r_c^k - r_1^k\| = \|r_c^k - r_3^k\|$. For a symmetric formation, $\Delta r_x = \Delta r_y$, which can be considered as a constant value. Substituting the values of Δr_x and Δr_y into Equation (6), we get

$$\nabla^2 z(r_c,t) \approx \frac{z(r_1^k,k) + z(r_2^k,k) + z(r_3^k,k) + z(r_4^k,k) - 4z(r_c^k,k)}{\Delta r_x^2}, \tag{7}$$

where $z(r_i^k,k)$ for $i = 1,2,3,4$ are field values at positions r_i^k at time step k. Applying the finite differences (6), (7) to

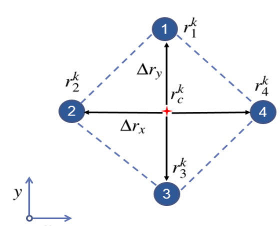


Fig. 1. Formation of four agents

Equation (1) gives:

$$\frac{z(r_c^k,k+1) - z(r_c^k,k)}{t_s} = \theta \cdot \frac{\sum_{i=1}^4 z(r_i^k,k) - 4z(r_c^k,k)}{\Delta r_x^2}. \tag{8}$$

If we can get the estimates of $z(r_c^k,k+1)$, $z(r_c^k,k)$, and $\sum_{i=1}^4 z(r_i^k,k)$, then θ can be estimated using RLS based on the discretized model (8). It should be noted that the sampling time t_s must obey the inequality $t_s \le \frac{\Delta r_x^2 + \Delta r_y^2}{4\theta}$ for the discretization method to converge [17].

Remark III.1 *Even though we only consider four agents in the above derivative, it should be noted that our scheme can be readily extended to $N > 4$ case by following the implementation of standard finite-volume method.*

B. The design of the cooperative Kalman filter

We first introduce the motivation of designing a cooperative filter by pointing out the difference between $z(r_c^k, k+1)$ and $z(r_c^k, k)$ in (5). By designing a cooperative filter similar to the one developed in [10], $z(r_c^k, k)$ may be directly estimated by combining the measurements taken by the sensing agents at time step k. However, at time step $k+1$, the formation center of the group is at position r_c^{k+1}. Therefore, the same filter can only provide the estimate of $z(r_c^{k+1}, k+1)$, not $z(r_c^k, k+1)$. In order to estimate the temporal variations of the field value in (5) along the trajectory of the mobile sensor network, we first need to derive a new cooperative filter to estimate both $z(r_c^k, k)$ and $z(r_c^k, k+1)$.

To construct a cooperative Kalman filter to obtain the estimates of $z(r_c^k, k)$ and $z(r_c^k, k+1)$, we first analyze the dynamics of the diffusion field value along the trajectory of the formation center r_c according to

$$\dot{z}(r_c,t) = \frac{\partial z(r_c,t)}{\partial r_c}\frac{dr_c}{dt} + \frac{\partial z(r_c,t)}{\partial t} = \nabla z(r_c,t)\cdot \dot{r}_c + \frac{\partial z(r_c,t)}{\partial t}, \tag{9}$$

where $\nabla z(r_c,t)$ is the gradient of $z(r_c,t)$. Substituting Equation (1) into (9), we obtain

$$\dot{z}(r_c,t) = \nabla z(r_c,t)\cdot \dot{r}_c + \theta \nabla^2 z(r_c,t). \tag{10}$$

Applying the finite differences to each term of (10) at time $t = t_{k-1}$ and at position $r_c = r_c^{k-1}$ gives:

$$\begin{aligned}
\dot{z}(r_c,t)|_{t=t_{k-1},r_c=r_c^{k-1}} &= \frac{z(r_c^k,k) - z(r_c^{k-1},k-1)}{t_s},\\
\nabla z(r_c,t)\dot{r}_c|_{t=t_{k-1},r_c=r_c^{k-1}} &= \frac{(r_c^k - r_c^{k-1})^T \nabla z(r_c^{k-1},k-1)}{t_s},\\
\theta\nabla^2 z(r_c,t)|_{t=t_{k-1},r_c=r_c^{k-1}} &= \theta\frac{\sum_{i=1}^4 z(r_i^{k-1},k-1) - 4z(r_c^{k-1},k-1)}{\Delta r_x^2}
\end{aligned} \tag{11}$$

Substituting the finite difference equations (11) to Equation (10) gives the information dynamics of $z(r_c^k, k)$ as

$$\begin{aligned}
z(r_c^k,k) = &\left(1 - \frac{4t_s\cdot\hat{\theta}_k}{\Delta r_x^2}\right) z(r_c^{k-1},k-1) + \frac{t_s\hat{\theta}_k}{\Delta r_x^2}\sum_{i=1}^4 z(r_i^{k-1},k-1)\\
&+ (r_c^k - r_c^{k-1})^T \nabla z(r_c^{k-1},k-1),
\end{aligned} \tag{12}$$

where $\hat{\theta}_k$ is the estimate of θ, which can be obtained from the RLS method that will be introduced in Section III-E. Similarly, we also obtain the dynamics of $z(r_c^k, k+1)$ by discretizing Equation (10) at time $t = t_k$ and at position $r_c = r_c^{k-1}$.

$$\begin{aligned}
z(r_c^k,k+1) = &\left(1 - \frac{4t_s\cdot\hat{\theta}_k}{\Delta r_x^2}\right) z(r_c^{k-1},k) + \frac{t_s\hat{\theta}_k}{\Delta r_x^2}\sum_{i=1}^4 z(r_i^k,k)\\
&+ (r_c^k - r_c^{k-1})^T \nabla z(r_c^{k-1},k).
\end{aligned} \tag{13}$$

Furthermore, we are also interested in estimating $\nabla z(r_c,t)$ since these estimates will not only provide insights of the structure of the field, but also be used in the motion control that will be introduced in Section III-F. We derive the total time derivative of $\nabla z(r_c,t)$ as

$$\dot{\nabla} z(r_c,t) = H(r_c,t)\cdot \dot{r}_c + \frac{\partial \nabla z(r_c,t)}{\partial t}, \tag{14}$$

where $H(r_c,t)$ is the Hessian matrix, and $\frac{\partial \nabla z(r_c,t)}{\partial t}$ is the higher order term, which can be considered as noise. By discetizing Equation (14) at $t = t_{k-1}, r_c = r_c^{k-1}$ and $t = t_k, r_c = r_c^{k-1}$, respectively, we can get that $\nabla z(r_c^k,k)$ and $\nabla z(r_c^k,k+1)$ evolve according to the following equations:

$$\begin{aligned}
\nabla z(r_c^k,k) &= \nabla z(r_c^{k-1},k-1) + H(r_c^{k-1},k-1)(r_c^k - r_c^{k-1}),\\
\nabla z(r_c^k,k+1) &= \nabla z(r_c^{k-1},k) + H(r_c^{k-1},k)(r_c^k - r_c^{k-1}).
\end{aligned} \tag{15}$$

Define the information state as $X(k+1) = [z(r_c^k,k), \nabla z(r_c^k,k), z(r_c^k,k+1), \nabla z(r_c^k,k+1)]^T$. By combing (12), (13), and the (15), the information state evolves according to the following equation:

$$X(k+1) = A_{\hat{\theta}}(k)X(k) + U(k) + w(k), \tag{16}$$

where $w(k)$ is the zero-mean process noise with covariance matrix Q, the matrices $A_{\hat{\theta}}(k)$ and $U(k)$ are defined by

$$A_{\hat{\theta}}(k) = \begin{bmatrix} 1 - \frac{4t_s\cdot\hat{\theta}_k}{\Delta r_x^2} & (r_c^k - r_c^{k-1})^T & 0 & 0\\ 0 & I_{2\times 2} & 0 & 0\\ 0 & 0 & 1 - \frac{4t_s\cdot\hat{\theta}_k}{\Delta r_x^2} & (r_c^k - r_c^{k-1})^T\\ 0 & 0 & 0 & I_{2\times 2} \end{bmatrix}, \tag{17}$$

$$U(k) = \begin{bmatrix} \frac{t_s\hat{\theta}_k}{\Delta r_x^2}\sum_{i=1}^4 z(r_i^{k-1},k-1)\\ H(r_c^{k-1},k-1)(r_c^k - r_c^{k-1})\\ \frac{t_s\hat{\theta}_k}{\Delta r_x^2}\sum_{i=1}^4 z(r_i^k,k)\\ H(r_c^{k-1},k)(r_c^k - r_c^{k-1}) \end{bmatrix}, \tag{18}$$

where $H(r_c^{k-1},k)$ is the Hessian matrix. We observe that $U(k)$ is determined by the estimates of $z(r_i^{k-1},k-1)$, $z(r_i^k,k)$ and the Hessian matrix, which will be specified in Section III-D.

A measurement equation is also required for the cooperative Kalman filter. Assuming r_i^k and r_i^{k-1} are close to r_c^{k-1}, the concentration can be locally approximated by a Taylor series up to second order as

$$\begin{aligned}
z(r_i^{k-1},k-1) \approx\ & z(r_c^{k-1},k-1) + (r_i^{k-1} - r_c^{k-1})^T \nabla z(r_c^{k-1},k-1)\\
&+ \frac{1}{2}(r_i^{k-1} - r_c^{k-1})^T H(r_c^{k-1},k-1)(r_i^{k-1} - r_c^{k-1}),\\
z(r_i^k,k) \approx\ & z(r_c^{k-1},k) + (r_i^k - r_c^{k-1})^T \nabla z(r_c^{k-1},k)\\
&+ \frac{1}{2}(r_i^k - r_c^{k-1})^T H(r_c^{k-1},k)(r_i^k - r_c^{k-1}).
\end{aligned} \tag{19}$$

Then, we denote the measurement vector as $P(k) = [p(r_1^{k-1},k-1)\cdots p(r_N^{k-1},k-1)\ \ p(r_1^k,k)\cdots p(r_N^k,k)]^T$, which is modeled as,

$$P(k) = C(k)\cdot X(k) + D(k)\vec{H}(k) + v(k), \tag{20}$$

where $\vec{H}(k) = [\vec{H}(r_c^{k-1},k-1)\ \ \vec{H}(r_c^{k-1},k)]^T$ represents the estimate of the Hessian at the center r_c^{k-1} in a vector form,

$v(k)$ is the zero-mean measurement noise with covariance matrix R.

$$C(k)=\begin{bmatrix} 1 & (r_1^{k-1}-r_c^{k-1})^T & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (r_N^{k-1}-r_c^{k-1})^T & 0 & 0 \\ 0 & 0 & 1 & (r_1^{k}-r_c^{k-1})^T \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & (r_N^{k}-r_c^{k-1})^T \end{bmatrix}, \tag{21}$$

and

$$D(k)=\begin{bmatrix} \frac{1}{2}((r_1^{k-1}-r_c^{k-1})\otimes(r_1^{k-1}-r_c^{k-1}))^T & 0 \\ \vdots & \vdots \\ \frac{1}{2}((r_N^{k-1}-r_c^{k-1})\otimes(r_N^{k-1}-r_c^{k-1}))^T & 0 \\ 0 & \frac{1}{2}((r_1^{k}-r_c^{k-1})\otimes(r_1^{k}-r_c^{k-1}))^T \\ \vdots & \vdots \\ 0 & \frac{1}{2}((r_N^{k}-r_c^{k-1})\otimes(r_N^{k}-r_c^{k-1}))^T \end{bmatrix}, \tag{22}$$

where $\otimes$ is the Kronecker product. In this paper, we assume the covariance matrices Q and R are known through measurements or computation [10].

Moreover, from Equation (8), we observe that there is a constraint between the state $z(r_c^k,k+1)$ and $z(r_c^k,k)$ at each step. Hence, we add a state equality constraint for the above Kalman filter:

$$G(k)\cdot X(k)=d(k), \tag{23}$$

where $G(k)=[1\quad 0\quad (\frac{4\cdot t_s\hat{\theta}_k}{\Delta r_x^2}-1)\quad 0]$ and $d(k)=\frac{t_s\hat{\theta}_k}{\Delta r_x^2}\sum_{i=1}^4 z(r_i^{k-1},k-1)$. We observe that the proposed Kalman filter is based on the time-varying information dynamics (16) with the state equality constraint (23). This type of Kalman filter with state equality constraint has been previously investigated in [18].

C. The cooperative Kalman filter equations

By following canonical procedures in [18], the equations for Kalman filter with state equality constraints are given as

(1) the one-step prediction,

$$\hat{X}^-(k)=A_{\hat{\theta}}(k-1)\tilde{X}^+(k-1)+U(k-1); \tag{24}$$

where $\tilde{X}^+(k-1)$ is the current constrained state estimate and $\hat{X}^-(k)$ is a prior unconstrained state estimate.

(2) error covariance for the one-step prediction,

$$P_c^-(k)=A_{\hat{\theta}}(k-1)P_c^+(k-1)A_{\hat{\theta}}^T(k-1)+Q; \tag{25}$$

(3) optimal gain,

$$K(k)=P_c^-(k)C^T(k)[C(k)P_c^-(k)C^T(k)+R]^{-1}; \tag{26}$$

(4) updated unconstrained estimate,

$$\hat{X}^+(k)=\hat{X}^-(k)+K(k)(P(k)-C(k)\hat{X}^-(k)-D(k)\vec{H}(k)); \tag{27}$$

(5) error covariance for the updated estimate

$$P_c^+(k)=(I-K(k)C(k))P_c^-(k); \tag{28}$$

(6) updated constrained estimate,

$$\begin{aligned}\tilde{X}^+(k)=\hat{X}^+(k)-P_c^+(k)G^T(k)[G(k)P_c^+(k)G^T(k)]^{-1}\\ \cdot\left(G(k)\hat{X}^+(k)-d(k)\right).\end{aligned} \tag{29}$$

D. Cooperative estimation of the Hessian

Estimates of $z(r_i^k,k)$, $z(r_i^{k-1},k-1)$, and the Hessian $\vec{H}(k)$ in Equation (18) are needed to enable the cooperative Kalman filter.

1) Estimates of $z(r_i^k,k)$, $z(r_i^{k-1},k-1)$: Since the sensor measurements $p(r_i^k,k)$ and $p(r_i^{k-1},k-1)$ are available in the measurement vector $P(k)$, one straightforward and simple way is to replace $z(r_i^k,k)$ and $z(r_i^{k-1},k-1)$ with the sensor measurements $p(r_i^k,k)$ and $p(r_i^{k-1},k-1)$.

2) Cooperative estimation of the Hessian: Using the cooperative Kalman filter, we can obtain a prediction for $X(k)$ as $\hat{X}^-(k)=A_{\hat{\theta}}(k-1)\tilde{X}^+(k-1)+U(k-1)$. If we assume the number of sensor $N\geq 4$ and the formation is not colinear, we have $P(k)=C(k)\cdot\hat{X}^-(k)+D(k)\vec{H}(k)$. The Hessian estimate can be solved by using the least mean square method, $\vec{H}(k)=\left(D(k)^T D(k)\right)^{-1}D(k)^T\left(P(k)-C(k)\hat{X}^-(k)\right)$.

E. Recursive Least Square Estimation

In this section, we use the RLS method to iteratively update the estimate of θ in the diffusion equation (1). We do this using the information state $X(k+1)=[z(r_c^k,k),\nabla z(r_c^k,k),z(r_c^k,k+1),\nabla z(r_c^k,k+1)]^T$ obtained from the filter to calculate the temporal variations of the field value along the trajectory $\frac{z(r_c^k,k+1)-z(r_c^k,k)}{t_s}$ at each step. On the other hand, by replacing $\sum_{i=1}^4 z(r_i^k,k)$ with the sensor measurements $\sum_{i=1}^4 p(r_i^k,k)$, we can estimate the discrete Laplacian operator directly using sensor measurements $P(k)$ from the agents according to Equation (7). Therefore, the diffusion coefficient can be directly estimated without the need of numerically solving the diffusion equation. Given an initial estimate for the diffusion coefficient, a simple application of the RLS method can iteratively update the estimate of θ. Following the canonical procedure of RLS estimation outlined in [19], we derive the following equations

$$\hat{\theta}_k=\hat{\theta}_{k-1}+g(k)\left(\frac{z(r_c^k,k+1)-z(r_c^k,k)}{t_s}-\nabla^2 z(r_c^k,k)\hat{\theta}_{k-1}\right),$$

$$g(k)=\eta(k-1)\nabla^2 z(r_c^k,k)^T\left[\nabla^2 z(r_c^k,k)\eta(k-1)\nabla^2 z(r_c^k,k)^T+R_e\right]^{-1},$$

$$\eta(k)=\left(I-g(k)\nabla^2 z(r_c^k,k)\right)\eta(k-1).$$

where $g(k)$ is the estimator gain matrix, $\eta(k)$ is the estimation error covariance matrix, and R_e is the noise covariance.

F. Formation and motion control

Control laws for the velocities of the agents are required so that the mobile sensor network can move along a certain trajectory while maintaining a desired formation. We view the entire formation as a deformable body. Thus, there are two parts of control: motion control and formation control. With the gradient estimates provided by the cooperative Kalman filter, the motion control for the agents can be easily realized by setting the velocities of the agents to be aligned with the estimated gradient direction. Thus, the mobile sensor network can achieve simultaneously parameter estimation and gradient climbing. Furthermore, there exists several results about the formation control for mobile agents

[10] [20]. We omit the detailed design of control here due to space limitation. Interested readers can refer to [10] [20].

IV. Convergence analysis of the cooperative Kalman filter

In this section, we prove the convergence of the cooperative Kalman filter. Theorem 7.4 in [21] states that if the time-varying system dynamics are uniformly complete controllable and uniformly complete observable, the Kalman filter for this system converges. With this result, we will establish a set of sufficient conditions for the mobile sensors such that the uniformly complete controllability and observability of the Kalman filter can be guaranteed. Let $\Phi(k,j)$ be the state transition matrix from time t_j to t_k, where $k>j$. Then, $\Phi(k,j)=A_{\hat{\theta}}(k-1)A_{\hat{\theta}}(k-2)\cdots A_{\hat{\theta}}(j)$ and $\Phi(k,j)=\Phi^{-1}(j,k)$. We have the following lemma.

Lemma IV.1 *For $\Phi(k,j)$ as defined above and $C(k)$ as defined in (21), we can have*

$$\Phi(k,j)=\begin{bmatrix} \xi_{\hat{\theta}} & (r_c^{k-1}-r_c^{j-1})^T & 0 & 0 \\ 0 & I_{2\times 2} & 0 & 0 \\ 0 & 0 & \xi_{\hat{\theta}} & (r_c^{k-1}-r_c^{j-1})^T \\ 0 & 0 & 0 & I_{2\times 2} \end{bmatrix}, \quad (30)$$

and

$$C(k)\Phi(k,j)=\begin{bmatrix} \xi_{\hat{\theta}} & (r_1^{k-1}-r_c^{j-1})^T & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_{\hat{\theta}} & (r_N^{k-1}-r_c^{j-1})^T & 0 & 0 \\ 0 & 0 & \xi_{\hat{\theta}} & (r_1^{k}-r_c^{j-1})^T \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \xi_{\hat{\theta}} & (r_N^{k}-r_c^{j-1})^T \end{bmatrix}, \quad (31)$$

where $\xi_{\hat{\theta}}=(1-\frac{4\cdot t_s}{\Delta r_x^2}\hat{\theta}_{k-1})(1-\frac{4\cdot t_s}{\Delta r_x^2}\hat{\theta}_{k-2})\cdots(1-\frac{4\cdot t_s}{\Delta r_x^2}\hat{\theta}_{j})$.

Let's first restate the definitions of uniformly complete controllability and uniformly complete observability, respectively (modified from Definitions in [21]).

Definition IV.2 *The proposed cooperative filter is uniformly complete controllable if there exist $\tau_1>0$, $\lambda_1>0$, and $\lambda_2>0$ such that the controllability Grammian $\mathfrak{C}(k,k-\tau_1)=\sum_{j=k-\tau_1}^{k}\Phi(k,j)Q\Phi(k,j)^T$ satisfies $\lambda_1 I_{6\times 6}\le\mathfrak{C}(k,k-\tau_1)\le\lambda_2 I_{6\times 6}$ for all $k>\tau_1$. Q is the covariance for the noise $w(k)$.*

In the following procedures, there exist some positive real numbers $\lambda_1,\lambda_2,\cdots\lambda_{20}$. All of these real numbers are time-independent bounds for various quantities, the values of which do not affect the correctness of our discussions. Note that, in this paper, a relation between two symmetric matrices $A_1\le A_2$ means that for any vector s with compatible dimension, there exists $s^TA_1s\le s^TA_2s$. We have the following lemma for uniformly complete controllability.

Lemma IV.3 *The proposed filter is uniformly complete controllable if the following conditions are satisfied:*

(Cd1) The covariance matrix Q is bounded, i.e., $\lambda_3 I\le Q\le\lambda_4 I$ for some constants $\lambda_3,\lambda_4>0$.

(Cd2) The speed of each agent is uniformly bounded, i.e., $\|r_i^j-r_i^{j-1}\|\le\lambda_5$ for all time j, for $i=1,\cdots,N$, and for some constant $\lambda_5>0$.

(Cd3) The estimated parameter $\hat{\theta}_j$ is bounded, i.e., $0\le\hat{\theta}_j<\lambda_6$. By properly selecting the sampling interval t_s and formation size Δr_x, we can make $\hat{\theta}_j$ satisfy that $0<1-\frac{4\cdot t_s}{\Delta r_x^2}\hat{\theta}_j\le 1$ for all time j, which means $\lambda_6<\frac{\Delta r_x^2}{4\cdot t_s}$.

Proof: Based on condition (Cd1), we obtain that the controllability Grammian satisfies $\lambda_3\sum_{j=k-\tau_1}^{k}\Phi(k,j)\Phi(k,j)^T\le\mathfrak{C}(k,k-\tau_1)$ and $\mathfrak{C}(k,k-\tau_1)\le\lambda_4\sum_{j=k-\tau_1}^{k}\Phi(k,j)\Phi(k,j)^T$ for any k and τ_1 such that $k>\tau_1$. Therefore, if we can find the uniform bounds for each of these semi-definite symmetric matrices, i.e., $\Phi(k,j)\Phi(k,j)^T$, the overall bound for the controllability Grammian can be obtained readily. We apply Lemma IV.1 to compute $\Phi(k,j)\Phi(k,j)^T$, i.e.,

$$\Phi(k,j)\Phi(k,j)^T= \quad (32)$$
$$\begin{bmatrix} \xi_{\hat{\theta}}+\|\delta r(k,j)\|^2 & (\delta r(k,j))^T & 0 & 0 \\ \delta r(k,j) & I_{2\times 2} & 0 & 0 \\ 0 & 0 & \xi_{\hat{\theta}}+\|\delta r(k,j)\|^2 & (\delta r(k,j))^T \\ 0 & 0 & \delta r(k,j) & I_{2\times 2} \end{bmatrix},$$

where we denote $\delta r(k,j)=r_c^{k-1}-r_c^{j-1}$.

Using basic linear algebra, we can obtain the minimum eigenvalue of matrix (32) as $\lambda_{min}=\frac{1}{2}\left(\|\delta r(k,j)\|^2+\xi_{\hat{\theta}}+1-\sqrt{\|\delta r(k,j)\|^2+\xi_{\hat{\theta}}+1)^2-4\cdot\xi_{\hat{\theta}}}\right)$, and the maximum eigenvalue as $\lambda_{max}=\frac{1}{2}\left(\|\delta r(k,j)\|^2+\xi_{\hat{\theta}}+1+\sqrt{\|\delta r(k,j)\|^2+\xi_{\hat{\theta}}+1)^2-4\cdot\xi_{\hat{\theta}}}\right)$. By definition, we know that $\delta r(k,j)$ is the averaged movement over all agents between time j and k. Then, the condition (Cd2) indicates that $\|\delta r(k,j)\|\le(k-j)\lambda_5\le\tau_1\lambda_5$ for all $j\in[k-\tau_1,k]$. Due to condition (Cd3), we can see that $0<\xi_{\hat{\theta}}\le 1$. Since the term $(\|\delta r(k,j)\|^2+\xi_{\hat{\theta}}+1)^2-4\cdot\xi_{\hat{\theta}}\ge(\xi_{\hat{\theta}}-1)^2\ge 0$, it can be proven readily that λ_{min} reaches its minimum value when $\|\delta r(k,j)\|=\tau_1\lambda_5$ and $\xi_{\hat{\theta}}$ approaches zero. That means the minimum value of λ_{min} is $\lambda_7>\frac{1}{2}(\|\tau_1\lambda_5\|^2+1-\sqrt{(\|\tau_1\lambda_5\|^2+1)^2})=0$. On the other hand, λ_{max} assumes its minimum value when $\|\delta r(k,j)\|=\tau_1\lambda_5$ and $\xi_{\hat{\theta}}$ is one. This maximum value is $\lambda_8=\frac{1}{2}(\|\tau_1\lambda_5\|^2+2+\sqrt{(\|\tau_1\lambda_5\|^2+2)^2})>0$. Therefore, we can conclude that $\lambda_7 I_{6\times 6}\le\Phi(k,j)\Phi(k,j)^T\le\lambda_8 I_{6\times 6}$ for all time $j\in[k-\tau_1,k]$. Hence, $\lambda_3\lambda_7\tau_1 I_{6\times 6}\le\mathfrak{C}(k,k-\tau_1)\le\lambda_4\lambda_8\tau_1 I_{6\times 6}$. Let $\lambda_1=\lambda_3\lambda_7\tau_1$ and $\lambda_2=\lambda_4\lambda_8\tau_1$. Thus, according to Definition IV.2, we have proved the uniformly complete controllability claim. ■

Definition IV.4 *The proposed cooperative filter is uniformly complete observable if there exist $\tau_2>0$, $\lambda_9>0$, and $\lambda_{10}>0$ such that the observability Grammian $\mathfrak{O}(k,k-\tau_2)=\sum_{j=k-\tau_2}^{k}\Phi^T(j,k)C^T(j)R^{-1}C(j)\Phi(j,k)$ satisfies $\lambda_9 I_{6\times 6}\le\mathfrak{O}(k,k-\tau_2)\le\lambda_{10}I_{6\times 6}$ for all $k>\tau_2$. Here R is the covariance for the measure noise $v(k)$.*

To prove the uniformly complete observability, we require one elementary Lemma IV.5 [10] and the Lemma IV.6 to establish the sufficient conditions for a moving formation.

Lemma IV.5 *Suppose two 2×1 vectors $a=[a_1\ a_2]^T$ and $b=[b_1\ b_2]^T$ form an angle Ψ such that $0<\Psi<\pi$. Then the* *minimum eigenvalue λ_{min} of the 2×2 matrix $M = a\cdot a^T + b\cdot b^T$ is strictly positive, i.e. $\lambda_{min} > 0$*

Lemma IV.6 *The proposed Kalman filter is uniformly completely observable if (Cd2), (Cd3) and the following conditions are satisfied:*

(Cd4) The number of agents N is greater than or equal 4.

(Cd5) The covariance matrix R is bounded, i.e., $\lambda_{11} I \leq R \leq \lambda_{12} I$ for some constants $\lambda_{11}, \lambda_{12} > 0$.

(Cd6) The distance between each agent and the formation center is uniformly bounded from both above and below, i.e., $\lambda_{13} \leq \|r_i^j - r_c^{j-1}\| \leq \lambda_{14}$ and $\lambda_{13} \leq \|r_i^{j-1} - r_c^{j-1}\| \leq \lambda_{14}$ for all j, for $i = 1,2,...,N$, and for some constants $\lambda_{13}, \lambda_{14} > 0$.

(Cd7) There exists a constant time difference τ_2, and for all $k > \tau_2$, there exists a time instance $j_2 \in [k-\tau_2, k]$, as well as two agents indexed by i_1 and i_2, such that the following condition is satisfied: The two vectors, $r_{i_1}^{j_2-1} - r_c^{j_2-1}$ and $r_{i_2}^{j_2-1} - r_c^{j_2-1}$ form an angle ψ_1 that is uniformly bounded away from 0 or π.

(Cd8) There exist the time instance j_2 and two agents i_1 and i_2 as given by (Cd7), such that the following condition is satisfied: The two vectors, $r_{i_1}^{j_2} - r_c^{j_2}$ and $r_{i_2}^{j_2} - r_c^{j_2-1}$ form an angle ψ_2 that is uniformly bounded away from 0 or π.

Proof: Based on condition (Cd5), we first observe that the observability Grammian satisfies $\lambda_{12}^{-1}\sum_{j=k-\tau_2}^{k}\Phi^T(j,k)C^T(j)C(j)\Phi(j,k) \leq \mathfrak{O}(k,k-\tau_2)$ and $\mathfrak{O}(k,k-\tau_2) \leq \lambda_{11}^{-1}\sum_{j=k-\tau_2}^{k}\Phi^T(j,k)C^T(j)C(j)\Phi(j,k)$ for any k and τ_2 such that $k > \tau_2$. Then the uniformly complete observability can be proved by finding the positive uniform upper and lower bounds for $\sum_{j=k-\tau_2}^{k}\Phi^T(j,k)C^T(j)C(j)\Phi(j,k)$ for all $k > \tau_2$. According to Lemma IV.1, we can get

$$\Phi^T(j,k)C^T(j)C(j)\Phi(j,k) = \begin{bmatrix} N\xi_{\hat\theta}^2 & N\xi_{\hat\theta}(r_c^{j-1}-r_c^{k-1})^T & 0 & 0 \\ N\xi_{\hat\theta}(r_c^{j-1}-r_c^{k-1}) & \sum_{i=1}^N (r_i^{j-1}-r_c^{k-1})(r_i^{j-1}-r_c^{k-1})^T & 0 & 0 \\ 0 & 0 & N\xi_{\hat\theta}^2 & N\xi_{\hat\theta}(r_c^{j}-r_c^{k-1})^T \\ 0 & 0 & N\xi_{\hat\theta}(r_c^{j}-r_c^{k-1}) & \sum_{i=1}^N (r_i^{j}-r_c^{k-1})(r_i^{j}-r_c^{k-1})^T \end{bmatrix}. \quad (33)$$

Due to conditions (Cd2) and (Cd6), we can observe that each element of the above matrix is bounded above, i.e., $\Phi(j,k)^T C(j)^T C(j)\Phi(j,k) \leq \lambda_{17}$ for some constant $\lambda_{17} > 0$. For the lower bound, we can use conditions (Cd4), (Cd7) and (Cd8) to prove that there exists the lower bound $\lambda_{18} > 0$ such that $\lambda_{18} \leq \sum_{j=k-\tau_2}^{k}\Phi(j,k)^T C(j)^T C(j)\Phi(j,k)$. Consider the two time instances indexed by $j_1, j_2 \in [k-\tau_2, k]$, where $j_1 < j_2$. If we can prove the matrix $\mathrm{I} = \Phi^T(j_1,k)C^T(j_1)C(j_1)\Phi(j_1,k) + \Phi^T(j_2,k)C^T(j_2)C(j_2)\Phi(j_2,k)$ has a lower bound, then $\sum_{j=k-\tau_2}^{k}\Phi(j,k)^T C(j)^T C(j)\Phi(j,k)$ has a lower bound. Since $\Phi(j_1,k) = \Phi(j_1,j_2)\Phi(j_2,k)$, we have

$$\begin{aligned} I &= [\Phi(j_1,j_2)\Phi(j_2,k)]^T C^T(j_1)C(j_1)\Phi(j_1,j_2)\Phi(j_2,k) \\ &+ \Phi^T(j_2,k)C^T(j_2)C(j_2)\Phi(j_2,k) \\ &= \Phi^T(j_2,k)[\Phi^T(j_1,j_2)C^T(j_1)C(j_1)\Phi(j_1,j_2) + C^T(j_2)C(j_2)]\Phi(j_2,k) \\ &= \Phi^T(j_2,k)I_1\Phi(j_2,k), \end{aligned} \quad (34)$$

where $I_1 = \Phi^T(j_1,j_2)C^T(j_1)C(j_1)\Phi(j_1,j_2) + C^T(j_2)C(j_2)$.

By using the fact that $\sum_{i=1}^N (r_i^{j_2-1} - r_c^{j_2-1}) = 0$, we have

$$C^T(j_2)C(j_2) = \begin{bmatrix} N & 0 & 0 & 0 \\ 0 & \sum_{i=1}^N (r_i^{j_2-1}-r_c^{j_2-1})(r_i^{j_2-1}-r_c^{j_2-1})^T & 0 & 0 \\ 0 & 0 & N & N(r_c^{j_2}-r_c^{j_2-1})^T \\ 0 & 0 & N(r_c^{j_2}-r_c^{j_2-1}) & \sum_{i=1}^N (r_i^{j_2}-r_c^{j_2})(r_i^{j_2}-r_c^{j_2})^T + N(r_c^{j_2}-r_c^{j_2-1})(r_c^{j_2}-r_c^{j_2-1})^T \end{bmatrix}, \quad (35)$$

$$\Phi^T(j_1,j_2)C^T(j_1)C(j_1)\Phi(j_1,j_2) = \begin{bmatrix} N\xi_{\hat\theta}^2 & N\xi_{\hat\theta}(r_c^{j_1-1}-r_c^{j_2-1})^T & 0 & 0 \\ N\xi_{\hat\theta}(r_c^{j_1-1}-r_c^{j_2-1}) & \sum_{i=1}^N (r_i^{j_1-1}-r_c^{j_2-1})(r_i^{j_1-1}-r_c^{j_2-1})^T & 0 & 0 \\ 0 & 0 & N\xi_{\hat\theta}^2 & N\xi_{\hat\theta}(r_c^{j_1}-r_c^{j_2-1})^T \\ 0 & 0 & N\xi_{\hat\theta}(r_c^{j_1}-r_c^{j_2-1}) & \sum_{i=1}^N (r_i^{j_1}-r_c^{j_2-1})(r_i^{j_1}-r_c^{j_2-1})^T \end{bmatrix}. \quad (36)$$

Then the matrix I can be obtained by combing (35) and (36) together. Consider two agents i_1 and i_2 in (Cd7) and (Cd8), we can further decompose I_1 as the sum of two matrices: a positive definite matrix I_2 and a positive semi-definite matrix I_3. Thus, the low bound of matrix I_1 can be proved by verifying that the matrix I_2 is positive definite and I_3 is positive semi-definite.

$$I_1 = I_2 + I_3 = \begin{bmatrix} N & 0 & 0 & 0 \\ 0 & I_4 & 0 & 0 \\ 0 & 0 & N & N(r_c^{j_2}-r_c^{j_2-1})^T \\ 0 & 0 & N(r_c^{j_2}-r_c^{j_2-1}) & I_5 \end{bmatrix} + I_3, \quad (37)$$

where $I_4 = (r_{i_1}^{j_2-1} - r_c^{j_2-1})(r_{i_1}^{j_2-1} - r_c^{j_2-1})^T + (r_{i_2}^{j_2-1} - r_c^{j_2-1})(r_{i_2}^{j_2-1} - r_c^{j_2-1})^T$, and $I_5 = N(r_c^{j_2} - r_c^{j_2-1})(r_c^{j_2} - r_c^{j_2-1})^T + (r_{i_1}^{j_2} - r_c^{j_2})(r_{i_1}^{j_2} - r_c^{j_2})^T + (r_{i_2}^{j_2} - r_c^{j_2-1})(r_{i_2}^{j_2} - r_c^{j_2-1})^T$. Because conditions (Cd7) and (Cd8) are satisfied, using Lemma IV.5, we can get that there exists $\lambda_{19} > 0$ such that $I_2 \geq \lambda_{19} I_{6\times 6}$. The residual term I_3 is

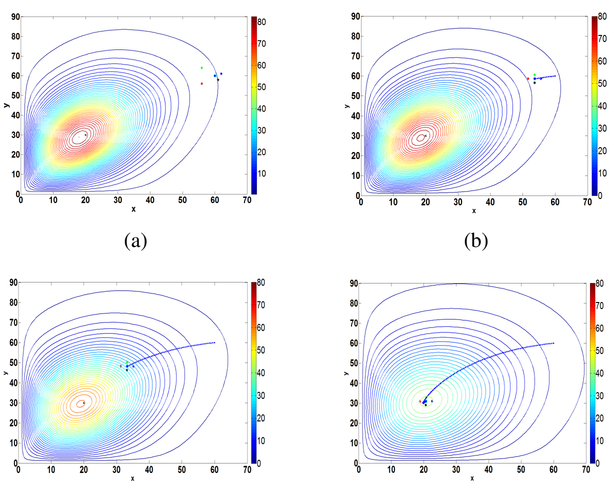
$$I_3 = \begin{bmatrix} N\xi_{\hat\theta}^2 & N\xi_{\hat\theta}(r_c^{j_1-1}-r_c^{j_2-1})^T & 0 & 0 \\ N\xi_{\hat\theta}(r_c^{j_1-1}-r_c^{j_2-1}) & I_6 & 0 & 0 \\ 0 & 0 & N\xi_{\hat\theta}^2 & N\xi_{\hat\theta}(r_c^{j_1}-r_c^{j_2-1})^T \\ 0 & 0 & N\xi_{\hat\theta}(r_c^{j_1}-r_c^{j_2-1}) & I_7 \end{bmatrix}, \quad (38)$$

where $I_6 = \sum_{i=1}^N (r_i^{j_2-1} - r_c^{j_2-1})(r_i^{j_2-1} - r_c^{j_2-1})^T - (r_{i_1}^{j_2-1} - r_c^{j_2-1})(r_{i_1}^{j_2-1} - r_c^{j_2-1})^T - (r_{i_2}^{j_2-1} - r_c^{j_2-1})(r_{i_2}^{j_2-1} - r_c^{j_2-1})^T + \sum_{i=1}^N (r_i^{j_1-1} - r_c^{j_1-1})(r_i^{j_1-1} - r_c^{j_1-1})^T + N(r_c^{j_1-1} - r_c^{j_2-1})(r_c^{j_1-1} - r_c^{j_2-1})^T$, and $I_7 = \sum_{i=1}^N (r_i^{j_1} - r_c^{j_1})(r_i^{j_1} - r_c^{j_1})^T + N(r_c^{j_1} - r_c^{j_2-1})(r_c^{j_1} - r_c^{j_2-1})^T - (r_{i_1}^{j_2} - r_c^{j_2})(r_{i_1}^{j_2} - r_c^{j_2})^T - (r_{i_2}^{j_2} - r_c^{j_2})(r_{i_2}^{j_2} - r_c^{j_2})^T$. For I_7, due to the formation control, we should have $\sum_{i=1}^N (r_i^{j_1} - r_c^{j_1})(r_i^{j_1} - r_c^{j_1})^T - \sum_{i=i_1,i_2} (r_i^{j_2} - r_c^{j_2})(r_i^{j_2} - r_c^{j_2})^T \geq 0$ when $N \geq 4$. Therefore, using Lemma IV.5, we can readily get that I_3 is a positive semi-definite matrix. The Weyl's theorem [10] can then be applied to prove that there exists

$\lambda_{20} > 0$ such that $I_1 \geq \lambda_{20} I_{6\times 6}$. Hence, $\mathfrak{O}(k, k-\tau_2)$ is uniformly bounded below by $\mathfrak{O}(k, k-\tau_2) \geq \lambda_9 I_{6\times 6}$. ■

V. Simulation Results

To demonstrate the performance of the proposed approach for online parameter estimation for diffusion processes, we consider a well-known 2D diffusion equation (1) with the nominal value of $\theta = 0.6$. The initial condition is illustrated in Fig. 2(a), in which the maximum value is at point (20,30). The whole domain of PDE is a rectangular area $0 \leq x \leq 70$, $0 \leq y \leq 90$. We implement an implicit ADI finite-difference scheme in MATLAB, with 100-by-100 spatial grid. We add 1% (in variance) white noise to measurements taken by sensors. A computational time step of $0.1s$ is chosen for the simulation, which satisfies the stability requirement of finite-difference method. In the simulation, we select the initial locations of four sensing agents represented by the red, blue, green, and purple stars as shown in Fig. 2(a). At each time step, the sensing agents take measurements of the field, run the proposed cooperative Kalman filter as well as the RLS algorithm to obtain the estimates of the diffusion coefficient θ, and move along the gradient direction estimated by the cooperative Kalman filter while converging to a desired formation. The trajectory of the center of the formation is shown in Fig. 2 (the blue dots). Initially, we set the estimate $\hat{\theta}_0 = 2$. The result of the proposed method is presented in Fig. 3. As we can observe from the figure, $\hat{\theta}$ converges to the nominal value.



(a) (b) (c) (d)

Fig. 2. Gradient climbing trajectory of the mobile sensor network.

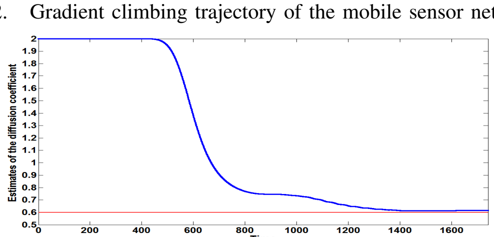


Fig. 3. Estimation of diffusion coefficient θ.

VI. Conclusion

We propose a novel filtering scheme for performing online parameter estimation for diffusion processes utilizing a mobile sensor network. Utilizing the filtered state as inputs, we employ the RLS algorithm to realize online estimation of the diffusion coefficient. Theoretical justifications are provided for the convergence analysis of the cooperative filter. Simulation results show satisfactory performance. Future work includes extending the proposed algorithm to spatially variant PDE models and demonstrating the methods in experiments involving robotic mobile sensor platforms.

References

[1] R. Ghez, *Diffusion Phenomena*. Kluwer Academic/ Plenum Publishers, 2nd edition, 2001.
[2] D. Uciński, *Optimal measurment methods for distributed parameter system identification*. CRC Press, 2004.
[3] S. Omatu and K. Matumoto, "Distributed parameter identification by regularization and its application to predication of air pollution," *Internation Journal of Systems Science*, vol. 22, no. 10, pp. 2001–2012, 1991.
[4] M. Krstic and A. Symshlyaev, "Adaptive control for PDEs," *Annual Reviews in Control*, vol. 32, no. 2, pp. 149–160, 2008.
[5] L. Rossi, B. Krishnamachari, and C. C. J. Kuo, "Distributed parameter estimation for monitoring diffusion phenomena using physical models," in *Sensor and Ad Hoc Communications and Networks, IEEE SECON, First Annual IEEE Communications Society Conference*, 2004, pp. 460–469.
[6] A. Symshlyaev and M. Krstic, "Adaptive boundary control for unstable parabolic PDEs - Part II: Estimation-based designs," *Automatica*, vol. 43, no. 9, pp. 1543–1556, 2007.
[7] W. Burgard, A. Cheyer, L. Stachniss, and F. E. Schneider, "Coordinated multi-robot exploration," *IEEE Transactions on Robotics*, vol. 21, no. 3, pp. 376–386, 2005.
[8] A. I. Mourikis and S. I. Roumeliotis, "Performance analysis of multirobot cooperative localization," *IEEE Transactions on Robotics*, vol. 22, no. 4, pp. 666–681, 2006.
[9] W. Wu and F. Zhang, "Robust cooperative exploration with a switching strategy," *IEEE Transactions on Robotics*, vol. 28, no. 4, pp. 828–839, 2012.
[10] F. Zhang and N. E. Leonard, "Cooperative control and filtering for cooperative exploration," *IEEE Transactions on Automatic Control*, vol. 55, no. 3, pp. 650–663, 2010.
[11] S. Martnez and F. Bullo, "Optimal sensor placement and motion coordination for target tracking," *Automatica*, vol. 42, no. 4, pp. 661–668, 2006.
[12] B. Grocholsky, J. Keller, V. Kumar, and G. Pappas, "Cooperative air and ground surveillance," *IEEE Robotics & Automation Magazine*, vol. 13, no. 3, pp. 16–25, 2006.
[13] Z. Tang and U. Ozguner, "Motion planning for multitarget surveillance with mobile sensor agents," *IEEE Transactions on Robotics*, vol. 21, no. 5, pp. 898–908, 2005.
[14] M. A. Demetriou, "Guidance of mobile actuator-plus-sensor networks for improved control and estaimation of distributed parameter system," *IEEE Transations on Automatic Control*, vol. 55, no. 7, pp. 1570–1584, 2010.
[15] D. Uciński and Y. Chen, "Time-optimal path planning of moving sensors for paramter estimation of distributed systems," in *Proc. of 44th IEEE Conf. on Decision and Control and European Control Conference*, 2005.
[16] E. Kalnay, *Atmospheric modeling, data assimilation, and predictability*. Cambridge University Press, 2003.
[17] M. A. Demetriou, N. A. Gatsonis, and J. R. Court, "Coupled control-computational fluids approach for the estimation of the concentration form a moving gaseous source in a 2-D domain with a lyapunov-guided sensing aerial vehicle," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 3, pp. 853–867, 2014.
[18] D. Simon and T. L. Chia, "Kalman filtering with state equality constraints," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 38, no. 1, pp. 128–136, 2002.
[19] L. Ljung, *System Identfication, 2nd ed.* Prentice-Hall, Englewood Cliffs, 1998.
[20] W. Ren and R. W. Beard, *Distributed consensus in multi-vehicle cooperative control, communications and control engineering series.* London: Springer-Verlag, 2008.
[21] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*. Academic Press, New York, 1970.