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Coordinated Orbit Transfer for Satellite Clusters[1]

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Abstract

We propose a control law which allows a satellite formation to achieve orbit transfer. During the transfer, the formation can be either maintained or modified to a desired one. Based on the orbit transfer control law proposed by Chang, Chichka and Marsden for single satellite, we add coupling terms to the summation of Lyapunov functions for single satellites. These terms are functions of the difference between the mean anomalies (or perigee passing times) of formation members. The asymptotic stability of the desired formation in desired orbits is proved.

1 Introduction

This paper is concerned with the problem of achieving a satellite formation near a designated elliptic orbit. For orbits near the earth, one can use a space shuttle to place satellites into specified relative positions. What we want to consider here is the case when members of the formation have been placed relatively far apart. They have to use their on-board thrusters to get to the desired orbits to form the desired formation. A similar case is when the whole formation has to be restructured for mission-related reasons. We want the formation to be maintained to some extent during the transfer and be re-established after the transfer.

Our approach is to use Lyapunov functions to design the control laws for orbit transfer. The Lyapunov function will achieve a local minimum when correct orbit and formation are reached. In [4], a Lyapunov function is expressed as a quadratic function of the differences of orbital elements between current orbit and the destination orbit. However, the convergence of the associated control law is not proved.

In this paper, we develop a new control algorithm and give a proof of convergence. This algorithm is based on a Lyapunov function on the shape space of elliptic orbits in our previous work [5], the work of Chang, Chichka and Marsden [1] and a result of Cushman and Bates [2]. However, the most significant extension is the addition of a coupling term which is a function of the difference between the osculating perigee passing times.

In section 2, we develop formulas used in the proofs of our theorems. In section 3, a brief summary of results in [1] and [5] are given.We introduce the definition of periodic satellite formations in section 4. Our main results and proofs about orbit transfer of periodic formations are presented in section 5. Simulation results are shown in section 6.

2 Preparations

If the mass of a satellite is small compared to the mass of the earth, the Kepler two body problem can be approximated by a one center problem as:

$$m\ddot{q} = -\nabla V_G + u \qquad (1)$$

where $q \in \mathcal{R}^3$ is the position vector of the satellite relative to the center of the earth, m is the mass of the satellite, V_G is the gravitational potential of the earth, u is the control force plus other disturbances. Without considering higher order terms, V_G takes the form

$$V_G = -m\frac{\mu}{\| q \|} \qquad (2)$$

Let $p = m\dot{q}$ be the momentum vector of the satellite. **For simplicity we assume that all the satellites considered in this paper have unit mass.**

Let us make the following definitions:

$$\begin{aligned} l(t) &= q(t) \times p(t) \\ A(t) &= p(t) \times l(t) - \mu\frac{q(t)}{\| q(t) \|} \\ e(t) &= \frac{\| A(t) \|}{\mu} \\ a(t) &= \frac{h(t)^2}{\mu(1-e(t)^2)} \\ cos(E(t)) &= \frac{1}{e(t)}(1 - \frac{r(t)}{a(t)}) \end{aligned}$$

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$$M(t) = E(t) - e(t)sin(E(t)) \quad (3)$$

where $h(t) = \| l(t) \|$ and $r(t) = \| q(t) \|$. These formulas can be found in textbooks on celestial mechanics [3]. l is the angular momentum vector. A is called the Laplace vector. They are conserved if $u(t) = 0$. a is the length of the semi-major axis and e is the eccentricity. E is the eccentric anomaly and M is the mean anomaly. The last equation is Kepler's equation. When $e(t) = 0$, the eccentric anomaly $E(t)$ is defined to be $M(t)$. For now, we will assume that $e(t) \neq 0$.

Notice that these formulas are valid for all t and all the elements are differentiable on $\mathcal{R}^3 \times \mathcal{R}^3 - \{0\}$. So we can take derivative on both sides of equations (3). By using the property that l, A,a and e are conserved when $u(t) = 0$, we have

$$\dot{l}(t) = \frac{\partial l}{\partial p} u(t) \quad (4)$$

$$\dot{A}(t) = \frac{\partial A}{\partial p} u(t) \quad (5)$$

$$\dot{e}(t) = \frac{1}{\mu} (\frac{\partial A}{\partial p})^T \widehat{A} \cdot u(t) \quad (6)$$

where $\widehat{A}$ is the unit vector along the direction of A. We also have

$$\dot{a}(t) = \frac{2}{\mu(1-e^2)} (\frac{\partial l}{\partial p})^T l \cdot u(t) + \frac{2ae}{\mu(1-e^2)} (\frac{\partial A}{\partial p})^T \widehat{A} \cdot u(t) \quad (7)$$

One can verify that,

$$\dot{E} = \sqrt{\frac{\mu}{a}} \frac{1}{r} + \frac{cos(E)}{e\, sin(E)} \dot{e} - \frac{r}{a^2 e\, sin(E)} \dot{a} \quad (8)$$

$$\dot{M} = (1 - e\,cos(E)) \dot{E} - \dot{e}\, sin(E) \quad (9)$$

Combining this equation with equation (8),(7)and (6), we have

$$\dot{M} = n + \eta (\frac{\partial A}{\partial p})^T \widehat{A} \cdot u(t) + \xi (\frac{\partial l}{\partial p})^T l \cdot u(t) \quad (10)$$

where $n = \sqrt{\frac{\mu}{a^3}}$ and

$$\xi(l, A, E) = -\frac{2(1 - e\,cos(E))^2}{\mu a e (1-e^2) sin(E)}$$
$$\eta(l, A, E) = \frac{cos(E) - e}{\mu e\, sin(E)} - \frac{2(1 - e\,cos(E))^2}{\mu(1-e^2) sin(E)} \quad (11)$$

Notice that ξ and η will be ∞ if $sin(E) = 0$. In order to prevent this from happening in our control laws, we will turn off the control when $sin(E) = 0$.

3 Orbit transfer of single satellite

For a single satellite on an elliptic orbit, the set D of ordered pairs (l, A) is a subset of $\mathcal{R}^3 \times \mathcal{R}^3$ with Euclidean norm,

$$D = \{(l, A) \in R^3 \times R^3 \mid A \cdot l = 0, l \neq 0, \| A \| < m^2 \mu\} \quad (12)$$

Let $W = \frac{1}{2} m \| \dot{q} \|^2 + V_G$ be the total energy of the satellite. Let P be the set of all pairs (q, p) with Euclidean norm. Then we can define a set Σ_e as

$$\Sigma_e = \{(q, p) \in P \mid W(q, p) < 0, l \neq 0\} \quad (13)$$

By the definition of l and A, we have a mapping π : $\Sigma_e \to D$, $(q, p) \mapsto (l, A)$.

Theorem 3.1 (Chang-Chichka-Marsden) [1]*The following hold:*
1. Σ_e is the union of all elliptic Keplerian orbits.
2. $\pi(\Sigma_e) = D$ and $\Sigma_e = \pi^{-1}(D)$.
3. The fiber $\pi^{-1}((l, A))$ is a unique (oriented) elliptic Keplerian orbit for each $(l, A) \in D$. (see also, [2], page 58)

The mapping π is a continuous mapping because (l, A) are continuous with respect to (q, p).

Corollary 3.2 *$\pi^{-1}(K)$ is compact for any compact set* $K \subset D$ (c.f. [5])

To control the orbit transfer of a single satellite, one considers a Lyapunov function from [1]

$$V(q, p) = \frac{1}{2} (\| l - l_d \|^2 + \| A - A_d \|^2) \quad (14)$$

where (l_d, A_d) is the pair of the angular momentum vector and Laplace vector of the target elliptic(circular) orbit. If we let the control to be

$$u = -[(l - l_d) \times q + l \times (A - A_d) + ((A - A_d) \times p) \times q] \quad (15)$$

then $\dot{V} \leq 0$ along the trajectory of the closed loop system. The following theorem is proved:

Theorem 3.3 (Chang-Chichka-Marsden) *There exists $c > 0$ such that if $V(q_0, p_0) \leq c$, by applying the control law as in equation (15), the trajectory of the closed loop system starting at (q_0, p_0) will asymptotically converge to the target orbit $\pi^{-1}((l_d, A_d))$*

4 Periodic formation

Suppose we have a formation consisting of m satellites. Let O_j denote the orbit of the jth satellite. We can make the following definition:

Definition 4.1 *A formation is periodic when a_j, the length of semi-major axis of orbit O_j, satisfies $a_j = a > 0$ for all $j = 1, 2, ...m$*

This definition is valid since all the satellites in a periodic formation will have the same orbital period

$$T = \frac{2\pi\sqrt{a^3}}{\sqrt{\mu}} \tag{16}$$

Thus although the shape of the formation is varying, it is varying periodically. However, the differences between the perigee passing times, $(\tau_i - \tau_j)$, are constants. Because the mean anomaly is

$$M_i = n_i(t - \tau_i) \tag{17}$$

where $n_i = 2\pi/T$, then $(M_i - M_j)$ are constants. By specifying the values of $(\tau_i - \tau_j)$ or $(M_i - M_j)$ for all i and j, a periodic formation can be uniquely determined.

5 Control laws for orbit transfer of satellite formations

To set up a periodic formation of two satellites, one can control each satellite separately to transfer to its target orbit. However, this will not assure the correct values of $(\tau_i - \tau_j)$ or $(M_i - M_j)$. In order to do that, extra terms involving $(\tau_i - \tau_j)$ or $(M_i - M_j)$ should be added in the summation of the Lyapunov functions for single satellites. This extension will result in a cooperative orbit transfer of multiple satellites.

We introduce a variable Υ_i which is defined as

$$\begin{aligned} \Upsilon_i &= \sqrt{\frac{a_i^3}{a_d^3}} M_i, \; if \;\; E_i \in (-\epsilon, \pi + \epsilon) \\ \Upsilon_i &= \sqrt{\frac{a_i^3}{a_d^3}} (2\pi - M_i); if \; E_i \in (\pi - \epsilon, 2\pi + \epsilon) \end{aligned} \tag{18}$$

for $i = 1, 2$, where a_d is the common length of the semi-major axes for the destination orbits.

Here, the trouble of using different expressions for the cases $E_i \in (-\epsilon, \pi + \epsilon)$ and $E_i \in (\pi - \epsilon, 2\pi + \epsilon)$ is caused by the fact that $E_i \in S^1$ a circle. Two coordinate charts are required on S^1. Here we pick the charts to be

$$\begin{aligned} \psi_1 &: (-\epsilon, \pi + \epsilon) \rightarrow (-\epsilon, \pi + \epsilon) \; s.t. \; E_i \mapsto E_i \\ \psi_2 &: (\pi - \epsilon, 2\pi + \epsilon) \rightarrow (-\epsilon, \pi + \epsilon) \; s.t. E_i \mapsto (2\pi - E_i) \end{aligned} \tag{19}$$

Here, the value of ϵ is chosen so that the two satellites will always be in the same chart. Because in a satellite formation the angular separations between satellites are usually small, the value of ϵ is small.

For $E_i \in (-\epsilon, \pi + \epsilon)$, we have

$$\begin{aligned} \dot{\Upsilon}_i &= \sqrt{\frac{a_i^3}{a_d^3}} \dot{M}_i + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \dot{a}_i M_i \\ &= \sqrt{\frac{\mu}{a_d^3}} + \rho(l_i, A_i, E_i)(\frac{\partial A_i}{\partial p_i})^T \widehat{A}_i \cdot u_i(t) \\ &\quad + \zeta(l_i, A_i, E_i)(\frac{\partial l_i}{\partial p_i})^T l_i \cdot u_i(t) \end{aligned} \tag{20}$$

where

$$\begin{aligned} \zeta &= \sqrt{\frac{a_i^3}{a_d^3}} \xi(l_i, A_i, E_i) + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \frac{2}{\mu(1 - e_i^2)} M_i \\ \rho &= \sqrt{\frac{a_i^3}{a_d^3}} \eta(l_i, A_i, E_i) + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \frac{2a_i e_i}{\mu(1 - e_i^2)} M_i \end{aligned} \tag{21}$$

For $E_i \in (\pi - \epsilon, 2\pi + \epsilon)$,

$$\begin{aligned} \dot{\Upsilon}_i &= -\sqrt{\frac{a_i^3}{a_d^3}} \dot{M}_i + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \dot{a}_i (2\pi - M_i) \\ &= -\sqrt{\frac{\mu}{a_d^3}} + \rho(l_i, A_i, E_i)(\frac{\partial A_i}{\partial p_i})^T \widehat{A}_i \cdot u_i(t) \\ &\quad + \zeta(l_i, A_i, E_i)(\frac{\partial l_i}{\partial p_i})^T l_i \cdot u_i(t) \end{aligned} \tag{22}$$

where

$$\begin{aligned} \zeta &= -\sqrt{\frac{a_i^3}{a_d^3}} \xi(l_i, A_i, E_i) + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \frac{2}{\mu(1 - e_i^2)} (2\pi - M_i) \\ \rho &= -\sqrt{\frac{a_i^3}{a_d^3}} \eta(l_i, A_i, E_i) + \frac{3}{2}\sqrt{\frac{a_i}{a_d^3}} \frac{2a_i e_i}{\mu(1 - e_i^2)} (2\pi - M_i) \end{aligned} \tag{23}$$

Notice that we have terms that explicitly contain M_i. If not handled well,these terms will cause discontinuity in our control algorithm when the satellites enter a new chart. The reason for us to pick the particular charts (ψ_1, ψ_2) is to reduce the discontinuities in the derivatives of Υ_i caused by changing charts.

We will design a Lyapunov function on the phase space of the two satellites. This one function will have different expressions in different charts. The Lyapunov function is

$$\begin{aligned} V &= V_1 + V_2 + 4sin(\frac{\Upsilon_1 - \Upsilon_2 - \phi}{4})^2 \\ &\quad if \;\; E_i \in (-\epsilon, \pi + \epsilon) \\ V &= V_1 + V_2 + 4sin(\frac{\Upsilon_1 - \Upsilon_2 + \phi}{4})^2 \\ &\quad if \;\; E_i \in (\pi - \epsilon, 2\pi + \epsilon) \end{aligned} \tag{24}$$

where

$$\begin{aligned} V_1 &= \frac{1}{2}(\| l_1 - l_{d1} \|^2 + \| A_1 - A_{d1} \|^2) \\ V_2 &= \frac{1}{2}(\| l_2 - l_{d2} \|^2 + \| A_2 - A_{d2} \|^2) \end{aligned} \tag{25}$$

Here, (l_{d1}, A_{d_1}) and (l_{d2}, A_{d_2}) specify the orbits in a two-satellite periodic formation and ϕ specifies the desired $(M_1 - M_2)$ on these orbits.

We can calculate the derivative of V as

$$\dot{V} = \dot{V}_1 + \dot{V}_2 + sin(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2})(\dot{\Upsilon}_1 - \dot{\Upsilon}_2) \tag{26}$$

The choice of $-$ or $+$ depends on the value of E_i as in the definition of V. By the calculations performed in the single satellite case,

$$\begin{aligned} \dot{V}_i &= [(l_i - l_{di})\times q_i + l_i\times(A_i - A_{di}) \\ &+((A_i - A_{di})\times p_i)\times q_i]\cdot u_i \end{aligned} \tag{27}$$

for $i = 1, 2$. Thus

$$\begin{aligned} \dot{V} &= [(l_1 - l_{d1} + \zeta_1 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})l_1)\times q_1 + \\ & l_1\times(A_1 - A_{d1} + \rho_1 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})\widehat{A}_1) + \\ & ((A_1 - A_{d1} + \rho_1 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})\widehat{A}_1)\times p_1)\times q_1]\cdot u_1 \\ & +[(l_2 - l_{d2} - \zeta_2 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})l_2)\times q_2 + \\ & l_2\times(A_2 - A_{d2} - \rho_2 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})\widehat{A}_2) + \\ & ((A_2 - A_{d2} - \rho_2 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})\widehat{A}_2)\times p_2)\times q_2]\cdot u_2 \end{aligned} \tag{28}$$

In order to get $\dot{V} \leq 0$, we let

$$\begin{aligned} u_1 &= -sin^2(E_1)[(l_1 - l_{d1} + \zeta_1 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})l_1)\times q_1 + \\ & l_1\times(A_1 - A_{d1} + \rho_1 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})\widehat{A}_1) + \\ & ((A_1 - A_{d1} + \rho_1 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})\widehat{A}_1)\times p_1)\times q_1] \\ u_2 &= -sin^2(E_2)[(l_2 - l_{d2} - \zeta_2 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})l_2)\times q_2 + \\ & l_2\times(A_2 - A_{d2} - \rho_2 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})\widehat{A}_2) + \\ & ((A_2 - A_{d2} - \rho_2 sin(\tfrac{\Upsilon_1-\Upsilon_2\mp\phi}{2})\widehat{A}_2)\times p_2)\times q_2] \end{aligned} \tag{29}$$

Notice that the factors $sin^2(E_i)$ cancel the term $sin(E_i)$ in the denominators of ζ_i and ρ_i. This will result in a continuous control law which will be 0 when $E_i = 0, \pi$.

Let $z = (q_1, p_1, q_2, p_2)$. We now proceed to find the initial condition $z_0 = (q_1(0), p_1(0), q_2(2), p_2(0))$ for z s.t. the set

$$S_M = \{z|V(z) \leq V(z_0)\} \tag{30}$$

is a compact subset of $\Sigma_{e1} \times \Sigma_{e2} - \{z|A_1 = 0 \; or \; A_2 = 0\}$. This is a necessary step because we want to apply LaSalle's invariance principle to prove our main result.

Lemma 5.1 *Let*

$$c < min\{c_1, c_2\} \tag{31}$$

where

$$c_i = min\{\frac{1}{2} \| A_{d_i} \|^2, \frac{1}{2} \| l_{di} \|^2, \frac{1}{2}(\mu - \| A_{di} \|)^2\} \tag{32}$$

for $i = 1, 2$. Then the set

$$S_M = \{z|V(z) \leq c\} \tag{33}$$

is a compact subset of $\Sigma_{e1} \times \Sigma_{e2} - \{z|A_1 = 0 \; or \; A_2 = 0\}$. **Proof:** The first observation is that the set

$$S_1 = \{(q_1, p_1)|V(q_1, p_1) \leq c^*, c^* < c_1\} \tag{34}$$

is a subset of Σ_{e1} i.e. $S_1 \cap \Sigma_{e1} = S_1$.

In fact, c_1 is the supremum of $V_1(q_1, p_1)$ on the set $\Sigma_{e1} - \{(q_1, p_1)|A_1 = 0\}$. To see this, we solve a constrained maximization problem as below:

$$sup\{\frac{1}{2}(\| l_1 - l_{d1} \|^2 + \| A_1 - A_{d1} \|^2)\} \tag{35}$$

under the constraints

$$A_1 \cdot l_1 = 0 \;\; l_1 \neq 0 \;\; A_1 \neq 0 \;\; and \;\; \| A_1 \| < \mu \tag{36}$$

First, we need to calculate the supremum of the unconstrained maximization problem. It is easy to see that this value is ∞. Then, we need to calculate the minimum value subject to $l_1 = 0$. The result is $\frac{1}{2} \| l_d \|^2$ achieved when $A = A_d$. Similarly, the minimum value subject to $A_1 = 0$ is $\frac{1}{2} \| A_d \|^2$. We should also calculate the minimum value subject to $\| A \| = \mu$. By applying the Lagrange multiplier method we found this value to be $\frac{1}{2}(\mu - \| A_d \|)^2$. Thus

$$c_1 = min\{\frac{1}{2} \| A_{d_1} \|^2, \frac{1}{2} \| l_{d1} \|^2, \frac{1}{2}(\mu - \| A_{d1} \|)^2\} \tag{37}$$

is the supremum of V_1 on the set $\Sigma_{e1} - \{(q_1, p_1)|A_1 = 0\}$. We proved that $S_1 \subset \Sigma_{e1} - \{(q_1, p_1)|A_1 = 0\}$.

Another observation is that $\pi(S_1)$ is a compact subset of D_1. Thus by corollary 3.2, S_1 is a compact subset of Σ_{e1}.

We can make the same arguments for the case when $i = 2$ to prove that S_2 is a compact subset of $\Sigma_{e2} - \{(q_2, p_2)|A_2 = 0\}$.

Hence by letting $c < min\{c_1, c_2\}$, it is true that

$$S_M \subset S_1 \times S_2 \subset \Sigma_{e1} \times \Sigma_{e2} - \{z|A_1 = 0 \; or \; A_2 = 0\} \tag{38}$$

Thus, S_M is a compact subset of $\Sigma_{e1} \times \Sigma_{e2} - \{z|A_1 = 0 \; or \; A_2 = 0\}$ ∎

We can now apply LaSalle's invariance principle to show that the trajectory of the closed loop system, starting within S_M, converges to the maximal invariant subset of S_M where $u(t) = 0$ is satisfied for all t.

Proposition 5.2 *With V,c and u_i given as in (24),(31) and (29), the trajectory starting from point $(q_{10}, p_{10}, q_{20}, p_{20})$ which satisfies*

$$V(q_{10}, p_{10}, q_{20}, p_{20}) \leq c \tag{39}$$

will converge to the set where

$$\begin{aligned} l_i &= l_{di} \\ A_i &= A_{di} \\ (M_1 - M_2) &= \phi \end{aligned} \tag{40}$$

are satisfied for $i = 1, 2$.

Proof: In order to calculate the invariant set,let $u_1 = 0$. When $sin(E_1) \neq 0$, we get

$$\begin{aligned}&(l_1 - l_{d1} + \zeta_1 sin(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2})l_1)\times q_1 \\ &+l_1\times(A_1 - A_{d1} + \rho_1 sin(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2})\widehat{A}_1) \\ &+((A_1 - A_{d1} + \rho_1 sin(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2})\widehat{A}_1)\times p_1)\times q_1 \\ &= 0 \end{aligned} \quad (41)$$

Take inner products on both sides with $q_1(t)$ to get

$$(l_1\times(A_1 - A_{d1} + \zeta_1 sin(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2})\widehat{A}_1))\cdot q_1 = 0 \quad (42)$$

This is equivalent to

$$(l_1 \times q_1(t)) \cdot (A_1 - A_{d1} + \zeta_1 sin(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2})\widehat{A}_1) = 0 \quad (43)$$

Let

$$B = A_1 - A_{d1} + \zeta_1 sin(\frac{\Upsilon_1 - \Upsilon_2 \mp \phi}{2})\widehat{A}_1 \quad (44)$$

Equation (43) means B is perpendicular to the vector $l_1 \times q_1(t)$. We can see that vector B should stay in the plane spanned by l_1 and $q_1(t)$.

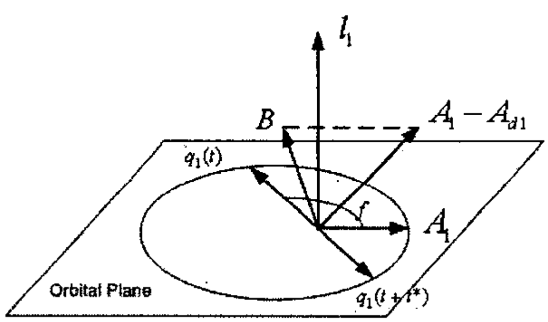

Figure 1: The relationship between l_1,q_1, $A_1 - A_d$ and B

However, from the assumption that $u_1(t) = 0$, we know that A_1 and l_1 are constant vectors. The time varying vector B will sweep a line segment passing the fixed point $(A_1 - A_{d1})$. The direction of this line segment is aligned with $\widehat{A}_1$. So vector B will be the intersection of the (l_1, q_1) plane and this line segment. Because $q_1(t)$ is sweeping the orbital plane, the (l_1, q_1) plane is identical at t and $t+kT_1$ where k is an integer and T_1 is the period of the first satellite. Since the line segment is not changed, the intersection points in these cases must be identical. Thus we must have

$$B(t) = B(t + kT_1) \quad (45)$$

Without lost of generality, suppose at time t, $E_1(t), E_2(t) \in (-\epsilon, \pi + \epsilon)$. Then equation (45) requires that

$$\begin{aligned}&\zeta_1(E_1(t))sin(\tfrac{\Upsilon_1(t)-\Upsilon_2(t)-\phi}{2}) \\ = \ &\zeta_1(E_1(t + kT_1))sin(\tfrac{\Upsilon_1(t+kT_1)-\Upsilon_2(t+kT_1)-\phi}{2})\end{aligned} \quad (46)$$

Let $k = 1$ in equation (46), because $\zeta_1(E_1(t)) = \zeta_1(E_1(t+T_1))$, the first observation we make is that the two satellites must have the same period. In fact, suppose at time t_0 equation (46) is satisfied. Then at time t_0+T_1, since $E_1(t_0) = E_1(t_0+T_1), \Upsilon_1(t_0) = \Upsilon_1(t_0+T_1)$ and all the angles(anomalies) are in the range of $[0, 2\pi)$, we must have $\Upsilon_2(t_0) = \Upsilon_2(t_0 + T_1)$. But

$$\Upsilon_2(t_0) - \Upsilon_2(t_0 + T_1) = -\sqrt{\frac{a_2^3}{a_d^3}}(M_2(t_0 + T_1) - M_2(t_0)) \quad (47)$$

Then $\Upsilon_2(t_0) = \Upsilon_2(t_0 + T_1)$ will be satisfied only if $M_2(t_0 + T_1) = M_2(t_0)$. Thus we shall have $T_1 = k_1T_2$ where k_1 is a positive integer. Remember we can apply the same argument to the second satellite to get $T_2 = k_1T_1$. Thus we must have $T_1 = T_2$. Hence on the invariant set, we proved that $a_1 = a_2$.

On the other hand, for a specific time t, we know that there exists $t^* \in [0, T_1)$ such that

$$\pi + f_1(t) = f_1(t + t^*) \quad (48)$$

where f_1 is the true anomaly of the first satellite. The value of t^* depends on t. The plane spanned by (l_1, q_1) at time t will also be identical to the plane spanned by (l_1, q_1) at time $t + t^*$. Thus we must have

$$B(t) = B(t + t^*) \quad (49)$$

which requires that

$$\begin{aligned}&\zeta_1(E_1(t))sin(\tfrac{\Upsilon_1(t)-\Upsilon_2(t)-\phi}{2}) \\ = \ &\zeta_1(E_1(t + t^*))sin(\tfrac{\Upsilon_1(t+t^*)-\Upsilon_2(t+t^*)+\phi}{2})\end{aligned} \quad (50)$$

Further, $a_1 = a_2$ implies that $M_1(t) - M_2(t) = M_1(t + t^*) - M_2(t + t^*)$, one can verify that

$$\begin{aligned}&sin(\tfrac{\Upsilon_1(t)-\Upsilon_2(t)-\phi}{2}) = \\ &-sin(\tfrac{\Upsilon_1(t+t^*)-\Upsilon_2(t+t^*)+\phi}{2})\end{aligned} \quad (51)$$

For (50) to be satisfied, one possibility is that

$$sin(\frac{\Upsilon_1(t) - \Upsilon_2(t) - \phi}{2}) = 0 \quad (52)$$

Another possibility is that

$$\zeta_1(E_1(t)) = -\zeta_1(E_1(t + t^*)) \quad (53)$$

By the definition of ζ_1, one can verify that (53) can only be satisfied when t takes value from a set of measure 0. Thus, for (50) to be satisfied, (52) must be true.

Because of (52), the time varying parts in equation (41) vanish. We can make the same argument as in the proof of the single satellite case [1] to show that

$$l_1 = l_{d1}$$

$$A_1 = A_{d1} \tag{54}$$

We can apply similar arguments for the second satellite. Thus we have

$$\begin{aligned} l_i &= l_{di} \\ A_i &= A_{di} \\ (\Upsilon_1 - \Upsilon_2) &= \pm\phi \end{aligned} \tag{55}$$

for $i = 1, 2$. By the definition of Υ_1 and Υ_2 in equation(18), we have

$$\sqrt{\frac{a_1^3}{a_d^3}}M_1 - \sqrt{\frac{a_2^3}{a_d^3}}M_2 = \phi \tag{56}$$

But we already know $a_1 = a_2 = a_d$, so we conclude that

$$(M_1 - M_2) = \phi \tag{57}$$

■

6 Simulation results

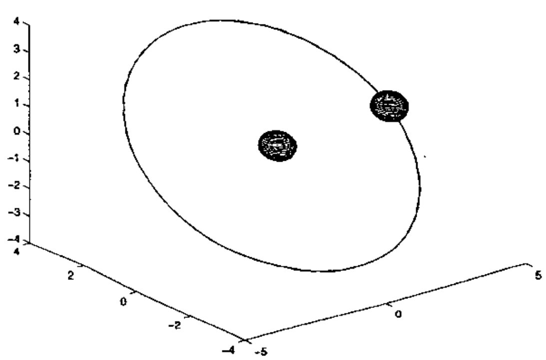

Figure 2: The desired final relative motion of two satellites (length unit= one tenth of earth radius)

To verify our algorithm, a series of simulations have been carried out. Here we will show a controlled transfer of two satellites from orbit $[a, e, i, \omega, \Omega] = [20, 0.1, \pi/4, \pi/2, 0]$ with initial separation of mean anomaly being $\pi/90$ to the orbit $[a, e, i, \omega, \Omega] = [25, 0.05, \pi/3, \pi/2, 0]$ with final separation of mean anomaly being $\pi/18$. Only relative motion between the satellites are plotted. Figure 2 displays the desired relative motion between the satellites. Figure 3 displays the relative motion between the satellites using the control algorithm proposed. As we can see, the desired orbit and separation are achieved.

7 Summary and future directions

In this paper we have proposed a control algorithm that can be used to set up periodic satellite formations on

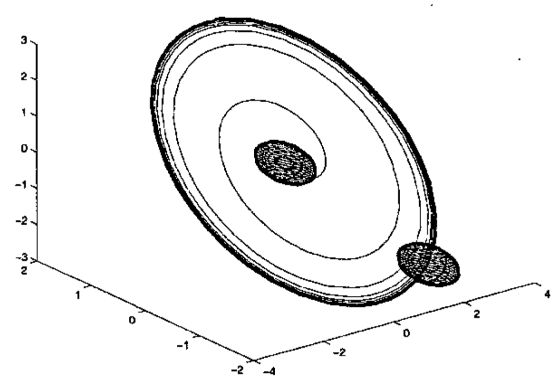

Figure 3: The relative motion achieved by our algorithm

elliptic orbits. The shape space formed by the angular momentum vectors and Laplace vectors is appropriate to describe satellite formations. The control laws we propose are based on a Lyapunov function on this shape space and proved to be convergent. We have not considered the effect of perturbations such as J_2 effect. This is currently being investigated.

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