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A Stochastic Optimization Framework for Source Seeking with Infotaxis-like Algorithms

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Abstract— **We propose a framework of algorithms for source seeking using stochastic optimization. We show that the infotaxis algorithm which uses information theory for source seeking can be realized using this framework. Using the framework, we developed a novel algorithm called the expected rate algorithm which has lower computational requirement. We prove that both infotaxis and expected rate algorithms generate identical optimization steps in most cases. Using simulation we show that under certain conditions the proposed algorithm generates more effective optimization steps than infotaxis and verify the computational performance of the proposed algorithm. We also demonstrate the practical applicability of the algorithm in source seeking through experiments.**

I. Introduction

Source seeking is typically described as a robot or a group of robots carrying sensors for measurement of a field, searching for the location of the source of the field. The field generated by the source can either be smooth or turbulent. A smooth field has a well defined gradient whereas the gradient of a turbulent field is not well-defined.

In the case of a smooth scalar field, methods involving gradient ascent/descent can be used to find the source location [1]–[6]. But, the problem becomes harder when the field is turbulent. Some algorithms have been proposed to deal with turbulent fields (eg. [7], [8]) with an assumption that measurement is available at all times but in a turbulent field the measurement is available only in sporadic manner. Infotaxis algorithm was proposed by Vergasolla et al. [9] which uses information entropy (a measure of spatial concentration of probability distribution of source location) for source localization. The infotaxis algorithm works reliably [10] but requires significant computation power.

This paper introduces a theoretical framework so that source seeking algorithms are formulated as iterative solutions of stochastic optimization problems. By changing the cost function used, one can derive various versions of the algorithms for both smooth and turbulent fields. This paper focuses on the case of turbulent fields and shows that the infotaxis algorithm can be explained by the framework. This paper also introduces an expected rate algorithm that has less computational demand than the infotaxis algorithm. The expected rate algorithm is analyzed and compared with the infotaxis algorithm. We show that the expected rate algorithm maximizes the inverse of expected entropy, while the infotaxis algorithm minimizes the expected change of entropy. We also show that in many situations the two algorithms produce the same optimization steps. Our simulation results also show that in certain situations the expected rate algorithm produces more aggressive optimization steps than the infotaxis algorithm. To our knowledge, these findings have not been documented in the literature. This paper also contains experimental results of the expected rate algorithm implemented on a Khepera robot to localize a light source.

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II. Problem formulation

Suppose a robot is trying to localize a source of turbulent plume field in a 2D space. The space is discretized into uniform grids. Let each grid point be denoted by h_i; $i = 1, 2, ..., N$ where N is the total number of grid points. The plume source has an unknown location represented by a random variable $\mathbf{Y}$. Let the robot's location be represented by the parameter θ and the measurement taken by the robot be represented by a random variable $\mathbf{Z}$.

A. Motion dynamics

We consider a simple particle model of the robot:

$$\theta_{k+1} = \theta_k + u_k \tag{1}$$

where θ_k is the current position of the robot at time $t = k$, u_k is the control input and θ_{k+1} is the next location.

B. Information dynamics

1) Probability distribution of source location: The random variable $\mathbf{Y}$ has a probability distribution $p(y)$. At $t = k$, given the measurement z_k at θ_k, we can update $p(y)$ using Bayes theorem as follows:

$$p(y|z_{1:k}, \theta_{1:k}) = \frac{p(y|z_{1:k-1}, \theta_{1:k})p(z_k|y, z_{1:k-1}, \theta_{1:k})}{p(z_k|z_{1:k-1}, \theta_{1:k})} \tag{2}$$

Assumption 2.1: Current measurement is dependent on the current location of the robot and the source location only.

$$p(z_k|y, z_{1:k-1}, \theta_{1:k}) = p(z_k|y, \theta_k) \tag{3}$$

Assumption 2.2: The probability distribution of the source location is independent of future robot location until the measurement is observed i.e.

$$p(y|z_{1:k-1}, \theta_{1:k}) = p(y|z_{1:k-1}, \theta_{1:k-1}) \tag{4}$$

Now using (2), (3) and (4) we have:

$$p(y|z_{1:k}, \theta_{1:k}) = \frac{p(y|z_{1:k-1}, \theta_{1:k-1})p(z_k|y, \theta_k)}{p(z_k|z_{1:k-1}, \theta_{1:k})} \tag{5}$$

Here $p(y|z_{1:k}, \theta_{1:k})$ is posterior, $p(y|z_{1:k-1}, \theta_{1:k-1})$ is prior, $p(z_k|y, \theta_k)$ is likelihood and $p(z_k|z_{1:k-1}, \theta_{1:k})$ is normalizer.

C. Measurement model

1) Hits: In a turbulent field, the true magnitude of the field far away from the source can either be zero or less than the threshold of the sensor. In both the cases, the robot does not detect the field. However, at some locations, the value of the field is above the sensor threshold and thus a detection is made by the sensor. These detections will be called *hits* in this paper and the number of hits observed by the robot is the measurement collected by it.

2) Rate of hits: The rate of hits is defined as the number of hits per unit time encountered by a sensor. Rate of hits is a function of the source location and the sensor (robot) location and since the source location is unknown therefore rate of hits $R(y, \theta)$ is also a random variable. A deterministic function of rate of hits for the case of plume particles in turbulent field can be derived using *Advection-diffusion equation* as was presented by Vergasolla et al. in [11].

3) Posterior distribution of source location: Given likelihood and nomalizer we can compute the posterior from (5).

Assumption 2.3: The number of hits received by a robot at any grid point is Poisson distributed and is independent of the number of hits received by it at any other grid point, given the source location.

Suppose the robot, searching a plume source stops at every visited location for time Δt to receive measurements. Using assumption 2.3, given the robot and source locations, probability of number of hits to be H_k can be calculated as:

$$p(z_k = H_k|y, \theta_k) = \frac{\exp(-R(y, \theta_k)\Delta t)(R(y, \theta_k)\Delta t)^{H_k}}{H_k!} \tag{6}$$

where $R(y, \theta_k)\Delta t$ is the Poisson rate parameter.
Similarly, the normalizer can be computed by marginalizing out random variable Y from the distribution $p(z_k, y|z_{1:k-1}, \theta_{1:k})$.

$$p(z_k|z_{1:k-1}, \theta_{1:k}) = \int p(z_k, y|z_{1:k-1}, \theta_{1:k})dy \tag{7}$$

Using (5), (6) and (7) we can calculate the posterior probability distribution of source location as follows:

$$\begin{aligned} p_k^+(y) &= p(y|z_{1:k}, \theta_{1:k}) \qquad (8)\\ &= \frac{p(y|z_{1:k-1}, \theta_{1:k-1}) \exp(-R(y, \theta_k)\Delta t)R(y, \theta_k)^{H_k}}{\int p(y|z_{1:k-1}, \theta_{1:k-1}) \exp(-R(y, \theta_k)\Delta t)R(y, \theta_k)^{H_k} dy} \end{aligned}$$

4) A priori distribution of the source location: At $t = k$, the a priori distribution of the source location $p_{k+1}^-(y)$ is an estimate of the probability distribution of source location if the search robot moves to $(\theta_k + u_k)$ and might receive $\hat{z}_{k+1}$ measurement after waiting for Δt time at that location. If, $p_k^+(y)$ is the posterior distribution of source location then using (8) the a priori distribution can be given as:

$$\begin{aligned} p_{k+1}^-(y) &= p(y|\hat{z}_{k+1}, z_{1:k}, \theta_k + u_k, \theta_{1:k}) \qquad (9)\\ &= \frac{p_k^+(y) \exp(-R(y, \theta_k + u_k)\Delta t)R(y, \theta_k + u_k)^{\hat{z}_{k+1}}}{\int p_k^+(y) \exp(-R(y, \theta_k + u_k)\Delta t)R(y, \theta_k + u_k)^{\hat{z}_{k+1}} dy} \end{aligned}$$

Algorithm 1: Template

Initialize search space with uniform probability distribution of source location.
Initialize iteration $k = 1$.
while $H_k < H_s$ **do**
    Let the posterior probability distribution be $p_k^+(y) = p(y|z_{1:k}, \theta_{1:k})$
    Compute the a priori probability distribution $p_{k+1}^-(y)$ using (9) $\forall z$
    Estimate the control input u_k using the following:

$$\arg\max_{u_k} \quad C_k(\theta_k, u_k, p_k^+(y), p_{k+1}^-(y))$$
$$\text{such that} \quad u_k = \gamma a_k;\ a_k \in D = \{\hat{\mathbf{0}}, \pm\hat{\mathbf{x}}, \pm\hat{\mathbf{y}}\}$$

    Move to the location θ_{k+1} using (1) and receive H_{k+1} hits after waiting for time Δt.
    Compute posterior distribution $p_{k+1}^+(y)$ using (8)
    $k = k + 1$
end

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5) Probability distribution of measurement: At time $t = k$, the distribution $p(\hat{z}_{k+1}|\theta_k + u_k)$ represents the probability of getting a measurement of $\hat{z}_{k+1}$ at a next possible location $(\theta_k + u_k)$ if the robot stays at that location for time Δt. We estimate this distribution by using assumption 2.3. If $\overline{R}_k^+(\theta_k + u_k) = \int p_k^+(y)R(y, \theta_k + u_k)dy$ is the expectation of rate of hits at location $(\theta_k + u_k)$ then the parameter of the Poisson process Λ can be estimated by the product of $\overline{R}_k^+(\theta_k + u_k)$ and the stopping time Δt. For $\overline{\theta} = \theta_k + u_k$:

$$p(\hat{z}_{k+1}|\overline{\theta}) = \frac{exp(-\overline{R}_k^+(\overline{\theta})\Delta t)(\overline{R}_k^+(\overline{\theta}))^{\hat{z}_{k+1}}}{\hat{z}_{k+1}!} \tag{10}$$

D. Source seeking

Let the plume source be located at S in the search space and $\delta \in \mathbb{R}^+$ be a small positive scalar such that the ball $B(S, \delta)$ is a neighborhood of the source. Let at $t = k$, the robot's position be θ_k and γ = grid-edge be the constant step size of robot. Let the posterior distribution of source location be $p_k^+(y) = p(y|z_{1:k}, \theta_{1:k})$ and the a priori distribution be $p_{k+1}^-(y) = p(y|\hat{z}_{k+1}, z_{1:k}, \theta_k + u_k, \theta_{1:k})$. Let $\hat{\mathbf{0}}$ be a vector of zero magnitude and $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ be unit vectors pointing in positive X and positive Y axes respectively. Let H_s be the upper bound on the number of hits received by a robot in time Δt. Given the above mentioned framework the next location of the robot can be computed by maximizing a cost function $C_k(\theta_k, u_k, p_k^+(y), p_{k+1}^-(y))$. The general framework for the source seeking algorithm can be given as algorithm 1.

It should be noted that we are restricting the movement of the robot along the edges of the grids of the discretized space. A source seeking algorithm generates a sequence of a control input u_k such that θ_k converges to the neighborhood of source location $B(S, \delta)$ as time goes to infinity i.e.

$$\lim_{k\to\infty} \quad \theta_k \in B(S, \delta) \tag{11}$$

Next, we discuss two types of source seeking algorithms under the framework of Algorithm 1. They differ by the cost function C_k. For convenience we assume $\gamma = 1$.

III. Infotaxis and expected rate algorithms

For the sake of simplicity, we define two expectation operators as follows:

$$E_1[f(.)] = \int p(\hat{z}_{k+1}|\theta_k + u_k) f(.) dz \quad (12)$$

$$E_2[f(.)] = \int p^-_{k+1}(y) f(.) dy \quad (13)$$

A. Infotaxis Algorithm

1) Information Entropy: The entropy of a probability distribution $p_k(y)$, can be given as the expectation of $(-\log p_k(y))$ with respect to $p_k(y)$. Hence, the entropy of the posterior distribution can be given as:

$$\overline{S}^+_k = -\int p^+_k(y) \log p^+_k(y) dy = E_Y[-\log p^+_k(y)] \quad (14)$$

2) Predicted entropy: Predicted entropy can be defined as the expected value of $(-\log p^-_{k+1}(y))$ with respect to the distribution $p(y, \hat{z}_{k+1}|z_{1:k}, \theta_k + u_k, \theta_{1:k})$.

$$\overline{S}^-_{k+1}(u_k, \theta_k) = E_1\left[E_2[-\log p^-_{k+1}(y)]\right] \quad (15)$$

3) Predicted change in entropy: Predicted change in the entropy at a new possible location of the robot $(\overline{\theta} = \theta_k + u_k)$ can be defined as:

$$\Delta\hat{S}_k(\overline{\theta}) = p^+_k(\overline{\theta})(-\overline{S}^+_k) + \left(1 - p^+_k(\overline{\theta})\right)\Delta S_{norm} \quad (16)$$
$$+ \left(1 - p^+_k(\overline{\theta})\right)\left(\overline{S}^-_{k+1}(\theta_k, u_k) - \sum_n p(\hat{z}_{k+1} = n|\overline{\theta})\overline{S}^+_k\right)$$

where ΔS_{norm} is the normalizer of the probability distribution when $p^+_k(\overline{\theta}) = 0$ and can be given as:

$$\Delta S_{norm} = -\sum_{h_u \neq \overline{\theta}} \frac{p^+_k(h_u)}{1 - p^+_k(\overline{\theta})} \log \frac{p^+_k(h_u)}{1 - p^+_k(\overline{\theta})} - \overline{S}^+_k \quad (17)$$

4) Algorithm: The infotaxis algorithm can be obtained by using negative of predicted change in entropy as cost function in algorithm 1 i.e. $C_k = -\Delta\hat{S}_k(u_k + \theta_k)$.

The first term of the cost function is an exploitation term and corresponds to the case when the source is found at $(\theta_k + u_k)$. This is based on a strong assumption that the robot can recognize the source if it is close to it.

The second and third terms are exploration terms which correspond to the case when the source is not found at $(\theta_k + u_k)$. The term ΔS_{norm} accounts for the fact that since source is not found at $(\theta_k + u_k)$, the probability distribution needs to be normalized and the term $(\overline{S}^-_{k+1}(\theta_k, u_k) - \sum_n p(\hat{z}_{k+1} = n|\theta_k + u_k)\overline{S}^+_k)$ accounts for the change in entropy introduced by the fact that the robot might receive some hits at $(\theta_k + u_k)$.

Remark 3.1: Estimation of $\Delta\hat{S}_k(\theta_k + u_k)$ involves estimation of three terms (equation (16)) which requires more computation compared to the expected rate algorithm.

B. Expected rate algorithm

1) Predicted rate of hits: The predicted rate of hits $\overline{R}^-_{k+1}(\theta_k, u_k)$ at a next possible robot location $(\theta_k + u_k)$ can be defined as the expected value of $R(y, \theta_k + u_k)$ with respect to the probability distribution $p(y|\theta_k + u_k)$.

$$\overline{R}^-_{k+1}(\theta_k, u_k) = E_1\left[E_2[R(y, \theta_k + u_k)]\right] \quad (18)$$

2) Algorithm: The expected rate algorithm can be obtained by using predicted rate of hits as cost function in algorithm 1 i.e. $C_k = \overline{R}^-_{k+1}(\theta_k, u_k)$.

Expected rate algorithm does not assume a measurement model which can recognize the source if the robot is close to it. We can show that estimation of cost function requires less computation when compared to infotaxis algorithm.

Let the total number of grid points be $N = l \times l$ and $p^+_k(y)$ be stored in a $l \times l$ matrix P. Let m be the total number of next possible steps in each iteration. The following are the additional computation, compared to expected rate algorithm, performed by the infotaxis algorithm for each iteration:

- Computation of S_k which requires taking log, multiplication and addition $\forall P_{i,j}$
- Dividing $(l^2 - 1)$ elements by $(1 - p^+_k(\theta_k))$, then taking their log, performing multiplication followed by addition to compute ΔS_{norm}.
- Taking log $\forall P_{i,j}$ a total of $3mH_s$ times to estimate predicted entropy $\forall m$.

Remark 3.2: Predicted rate of hits is weighted mean of rv $R(y, \theta)$ and since $R(y, \theta)$ decreases exponentially with an increase in distance between source location and robot location [11], therefore the value of predicted rate of hits can be found using approximate mean. The approximate mean can be calculated by ignoring the grid points where $R(y, \theta) \approx 0$. Hence the computation required can be reduced to a much lesser value.

IV. Relation between predicted entropy and predicted rate of hits

Lemma 4.1: Under the assumptions 2.1-2.3 and the framework of the algorithm 1, for some $u_k = u^1_k$, if

If,

$$R^+_k(\theta_k + u^1_k) E_2\left[\frac{1}{R(y, \theta_k + u^1_k)}\right] < 1 \quad (19)$$

then,

$$\overline{R}^-_{k+1}(\theta_k, u^1_k) \propto \frac{1}{f(\overline{S}^-_{k+1}(\theta_k, u^1_k))} \quad (20)$$

where $f(.)$ is some monotonic function.

Proof: Let $\overline{\theta} = (\theta_k + u^1_k)$. Predicted entropy at $\overline{\theta}$ is:

$$\overline{S}^-_{k+1}(\theta_k, u^1_k) = E_1\left[E_2[-\log p^-_{k+1}(y)]\right] \quad (21)$$

Using (9) we have

$$\overline{S}^-_{k+1}(\theta_k, u^1_k) =$$
$$-E_1\left[E_2\left[\log \frac{p^+_k(y)\exp(-R(y, \overline{\theta})\Delta t)R(y, \overline{\theta})^z}{\int p^+_k(y)\exp(-R(y, \overline{\theta})\Delta t)R(y, \overline{\theta})^z dy}\right]\right]$$

Now, after distribution of log, let us take the derivative of the predicted entropy with respect to $R(y, \overline{\theta})$

$$\frac{\partial \overline{S}^{-}_{k+1}(\theta_k, u^1_k)}{\partial R(y,\overline{\theta})} = -E_1\Bigg[E_2\Bigg[\frac{z}{R(y,\overline{\theta})} - \Delta t - \frac{\partial}{\partial R(y,\overline{\theta})} \log \int \exp(-R(y,\overline{\theta})\Delta t) R(y,\overline{\theta})^z dy\Bigg]\Bigg] \quad (22)$$

Now, using Jensen's inequality we have:

$$\frac{\partial \overline{S}^{-}_{k+1}(\theta_k, u^1_k)}{\partial R(y,\overline{\theta})} \leq -E_1\Bigg[E_2\Bigg[\frac{z}{R(y,\overline{\theta})} - \Delta t - \frac{\partial}{\partial R(y,\overline{\theta})} \int \log\big(\exp(-R(y,\overline{\theta})\Delta t) R(y,\overline{\theta})^z dy\big)\Bigg]\Bigg] \quad (23)$$

$$= -E_1\Bigg[E_2\Bigg[\frac{z}{R(y,\overline{\theta})} + \int \Delta t dy - \Delta t - \int \frac{z}{R(y,\overline{\theta})} dy\Bigg]\Bigg]$$

Since, E_1 and E_2 are linear operators and $R(y, \overline{\theta})$ does not depend on z therefore switching the positions of expectations and taking E_1 inside the RHS, we have:

$$= -E_2\Bigg[\frac{E_1[z]}{R(y,\overline{\theta})} + \int \Delta t dy - \Delta t - \int \frac{E_1[z]}{R(y,\overline{\theta})} dy\Bigg] \quad (24)$$

Again, taking expectation with respect to E_2 we have:

$$\frac{\partial \overline{S}^{-}_{k+1}(\theta_k, u^1_k)}{\partial R(y,\overline{\theta})} \leq -(N-1)\left(\Delta t - E_1[z] E_2\left[\frac{1}{R(y,\overline{\theta})}\right]\right)$$

From (10) we know that $\Delta t \overline{R}^{+}_k(\overline{\theta})$ is the Poisson rate parameter for the measurement probability i.e. $E_1[z] = \Delta t \overline{R}^{+}_k(\overline{\theta})$. Using this we have:

$$\frac{\partial \overline{S}^{-}_{k+1}(\theta_k, u^1_k)}{\partial R(y,\overline{\theta})} \leq -(N-1)\Delta t \left(1 - \overline{R}^{+}_k(\overline{\theta}) E_2\left[\frac{1}{R(y,\overline{\theta})}\right]\right) \quad (25)$$

We know that stopping time $\Delta t > 0$, thus for $N > 1$, Since $\overline{R}^{+}_k(\overline{\theta}) E_2\left[\frac{1}{R(y,\overline{\theta})}\right] < 1$, therefore we have:

$$\frac{\partial \overline{S}^{-}_{k+1}(\theta_k, u^1_k)}{\partial R(y,\overline{\theta})} < 0 \quad (26)$$

Also,

$$\frac{\partial \overline{R}^{-}_{k+1}(\theta_k, u^1_k)}{\partial R(y,\overline{\theta})} = \frac{\partial E_1\Big[E_2\Big[R(y,\overline{\theta})\Big]\Big]}{\partial R(y,\overline{\theta})} = 1 > 0 \quad (27)$$

Both predicted entropy $\overline{S}^{-}_{k+1}(\theta_k, u_k)$ and predicted rate of hits $\overline{R}^{-}_{k+1}(\theta_k, u_k)$ are implicitly dependent on $\overline{\theta}$ only through $R(y, \overline{\theta})$ therefore we have:

$$\frac{d\overline{S}^{-}_{k+1}(\theta_k, u^1_k)}{d\overline{\theta}} = \frac{\partial \overline{S}^{-}_{k+1}(\theta_k, u_k)}{\partial R(y,\overline{\theta})} \frac{dR(y,\overline{\theta})}{d\overline{\theta}} \quad (28)$$

$$\frac{d\overline{R}^{-}_{k+1}(\theta_k, u^1_k)}{d\overline{\theta}} = \frac{\partial \overline{R}^{-}_{k+1}(\theta_k, u_k)}{\partial R(y,\overline{\theta})} \frac{dR(y,\overline{\theta})}{d\overline{\theta}} \quad (29)$$

using (26), (27), (28) and (29) we can say that:

$$\overline{R}^{-}_{k+1}(\theta_k, u^1_k) \propto \frac{1}{f(\overline{S}^{-}_{k+1}(\theta_k, u^1_k))} \quad (30)$$

■

Remark 4.1: Lemma 4.1 implies that if we change the cost function in algorithm 1 from predicted rate of hits to inverse of the predicted entropy then the control inputs generated by the expected rate algorithm should be identical under the conditions given by the lemma.

Corollary 1.1: Under the framework of algorithm 1, given a monotonic function $f(.)$, for some $u_k = u^1_k$

$$\text{If, } E_2[R(y, \theta_k + u^1_k)] > \overline{R}^{+}_k(\theta_k + u^1_k)$$
$$\text{then, } \overline{R}^{-}_{k+1}(\theta_k, u^1_k) \propto \frac{1}{f(\overline{S}^{-}_{k+1}(\theta_k, u^1_k))} \quad (31)$$

Proof: Let us assume that:

$$\overline{R}^{+}_k(\theta_k + u^1_k) E_2\left[\frac{1}{R(y, \theta_k + u^1_k)}\right] < 1 \quad (32)$$

Let for simplicity $\overline{\theta} = (\theta_k + u^!_k)$ then from (32) we have:

$$\frac{1}{\overline{R}^{+}_k(\overline{\theta})} > E_2\left[\frac{1}{R(y,\overline{\theta})}\right] \quad (33)$$

Taking log of both sides and using Jensen's inequality we have:

$$\log \frac{1}{\overline{R}^{+}_k(\overline{\theta})} > E_2\left[\log \frac{1}{R(y,\overline{\theta})}\right] \quad (34)$$

Using the linearity of the expectation operator and Jensen's inequality we have:

$$\log E_2[R(y,\overline{\theta})] > \log \overline{R}^{+}_k(\overline{\theta}) \quad (35)$$

Thus, we have,

$$E_2[R(y,\overline{\theta})] > \overline{R}^{+}_k(\overline{\theta}) \quad (36)$$

Hence, using (32), (36) and lemma 4.1 we can say that (31) is satisfied. ■

Corollary 1.2: Under the framework of algorithm 1 and at $t = k$, if there exist a u^j_k such that $E_2[R(y, \theta_k + u^j_k)] > \overline{R}^{+}_k(\theta_k + u^j_k)$, then for any pair of next possible step of the robot $h_j = \theta_k + u^j_k$ and $h_m = \theta_k + u^m_k$, if $\overline{R}^{-}_{k+1}(\theta_k, u^j_k) > \overline{R}^{-}_{k+1}(\theta_k, u^m_k)$ then $\overline{S}^{-}_{k+1}(\theta_k, u^j_k) < \overline{S}^{-}_{k+1}(\theta_k, u^m_k)$.

Proof: Let at the current location θ_k, $\overline{R}^{-}_{k+1}(\theta_k, 0) = \overline{R}$ and $\overline{S}^{-}_{k+1}(\theta_k, 0) = \overline{S}$. Let for simplicity of notation $\overline{R}^{-}_{k+1}(\theta_k, u^i_k) = \overline{R}^{-}_{k+1}(h_i)$ and $\overline{S}^{-}_{k+1}(\theta_k, u^i_k) = \overline{S}^{-}_{k+1}(h_i)$ for $i = j, m$.

Given $E_2[R(y, \theta_k + u^j_k)] > \overline{R}^{+}_k(\theta_k + u^j_k)$, from lemma 4.1 and corollary 1.1, we can deduce that for the next possible location h_j we have $\overline{S}^{-}_{k+1}(h_j) < \overline{S}$. Now, the other option of next possible location h_m corresponds to one of the following three cases:

$$\text{Case 1: } \overline{R}^{+}_k(\theta_k + u^m_k) E_2[1/R(y, h_m)] > 1 \quad (37)$$

$$\text{Case 2: } \overline{R}^{+}_k(\theta_k + u^m_k) E_2[1/R(y, h_m)] \leq 1 \quad (38)$$

From (25) we can see that for Case 1, the gradient of $\overline{S}^{-}_{k+1}(h_m) > 0$. This means $\overline{S}^{-}_{k+1}(h_m) > \overline{S}$ or $\overline{S}^{-}_{k+1}(h_j) < \overline{S}^{-}_{k+1}(h_m)$. Again from (25) we can see that for Case 2, the gradient of $\overline{S}^{-}_{k+1}(h_m) \leq 0$ but using (28) and (29) we can say that since $\overline{R}^{-}_{k+1}(h_j) > \overline{R}^{-}_{k+1}(h_m)$ therefore $\overline{S}^{-}_{k+1}(h_j)$ has steeper gradient than $\overline{S}^{-}_{k+1}(h_m)$. Hence we have $\overline{S}^{-}_{k+1}(h_j) < \overline{S}^{-}_{k+1}(h_m)$. ■

Remark 4.2: Corollary 1.2 implies that if there exist one control input for which the average rate of hits increases then the control input which corresponds to the maximal increase in the predicted rate of hits also corresponds to the maximal decrease in the predicted entropy.

V. Performance of the Algorithms

Under the framework of algorithm 1, let at $t = k$, $h_j = \theta_k + u_k^j$ and $h_m = \theta_k + u_k^m$ be two of the choices for next location using the infotaxis algorithm and $\Delta^2 S_k$ be the difference of the predicted change in the entropy for h_j and h_m. Let for simplicity of notation $\overline{R}^{-}_{k+1}(\theta_k, u_k^i) = \overline{R}^{-}_{k+1}(h_i)$, $\overline{S}^{-}_{k+1}(\theta_k, u_k^i) = \overline{S}^{-}_{k+1}(h_i)$ and $p(\hat{z}_{k+1} = n|h_i) = \rho_{n,i}$ for $i = j, m$. Now, using (15), (16), (17) and after some grouping of the terms we obtain:

$$\begin{aligned} &\Delta^2 S_k = \Delta\hat{S}_k(h_m) - \Delta\hat{S}_k(h_j) = A + B + C \quad ; \\ &A = S_k(p_k^+(h_m) - p_k^+(h_j)) \\ &B = \log \frac{p_k^+(h_m)^{p_k^+(h_m)}(1 - p_k^+(h_m))^{(1-p_k^+(h_m))}}{p_k^+(h_j)^{p_k^+(h_j)}(1 - p_k^+(h_j))^{(1-p_k^+(h_j))}} \\ &C = (1 - p_k^+(h_m))\overline{S}^{-}_{k+1}(h_m) - (1 - p_k^+(h_j))\overline{S}^{-}_{k+1}(h_j) \end{aligned} \tag{39}$$

$\rho_{n,m}$ is the probability of getting n hits at location h_m therefore $\sum_n \rho_{n,m} = 1$. For source seeking in turbulent environment, number of hits are limited to a small number.

Assumption 5.1: Number of hits received by the robot at a location in time Δt is bounded by H_s such that $\sum_{n=1}^{H_s} \rho_{n,m} = \sum_{n=1}^{H_s} \rho_{n,j} = 1$.

Theorem 5.1: Using framework of algorithm 1, lets consider a robot using infotaxis for source seeking. Let at $t = k$, h_j and h_m be two of the next possible steps of the robot such that for $p_k^+(h_j) < 1/2$ & $p_k^+(h_m) < 1/2$, $p_k^+(h_j) = p_k^+(h_m) + a$; $a > 0$. Given $E_2[R(y, h_j)] > \overline{R}_k^+(h_j)$, if $\overline{R}^{-}_{k+1}(h_j) > \overline{R}^{-}_{k+1}(h_m)$ then the robot will move to h_j.

Proof: Using (39) we have:

$$\Delta^2 S_k = A + B + C; \quad \text{such that } A = aS_k > 0;$$

Function $g(x) = x^x(1-x)^{(1-x)}$ is a monotonically decreasing function for $0 \leq x \leq 0.5$. Since $p_k^+(h_j), p_k^+(h_m) < 0.5$ and $p_k^+(h_m) > p_k^+(h_j)$ therefore using (39) we have $B > 0$. Now, again using (39):

$$C = (1 - p_k^+(h_m))\overline{S}^{-}_{k+1}(h_m) - (1 - p_k^+(h_m) - a)\overline{S}^{-}_{k+1}(h_j) \tag{40}$$

Using corollary 1.2 we have if $\overline{R}^{-}_{k+1}(h_j) > \overline{R}^{-}_{k+1}(h_m)$ then $\overline{S}^{-}_{k+1}(h_m) > \overline{S}^{-}_{k+1}(h_j)$. Using (40), we can see that co-efficient of $\overline{S}^{-}_{k+1}(h_m)$ is greater than co-efficient of $\overline{S}^{-}_{k+1}(h_j)$ and since $\overline{S}^{-}_{k+1}(h_m) > \overline{S}^{-}_{k+1}(h_j)$ therefore we obtain $C > 0$. Since $A, B, C > 0$ therefore using (39) we can say have $\Delta\hat{S}_k(h_m) - \Delta\hat{S}_k(h_j) > 0$. Now, since infotaxis minimizes the predicted change in entropy therefore using infotaxis algorithm, robot will move to h_j. ■

Theorem 5.2: Using framework of algorithm 1, lets consider a robot using infotaxis for source seeking. Suppose at $t = k$, h_j and h_m be two of the next possible moves of the robot such that $p_k^+(h_m) \approx p_k^+(h_j)$. Given $E_2[R(y, h_j)] > \overline{R}_k^+(h_j)$, if robot moves to h_j then $\overline{R}^{-}_{k+1}(h_j) > \overline{R}^{-}_{k+1}(h_m)$.

Proof: Since, $p_k^+(h_m) \approx p_k^+(h_j)$ therefore we can say $p_k^+(h_m) = p_k^+(h_j) \pm \epsilon$; $\epsilon \ll 1$. Using (39) we have:

$$A = \pm\epsilon S_k; \; B = 0; \; C = (1 - p_k^+(h_j))[\overline{S}^{-}_{k+1}(h_m) - \overline{S}^{-}_{k+1}(h_j)]$$

such that

$$\Delta^2 S_k = \pm\epsilon S_k + (1 - p_k^+(h_j))[\overline{S}^{-}_{k+1}(h_m) - \overline{S}^{-}_{k+1}(h_j)] \tag{41}$$

If the robot moves to h_j using infotaxis algorithm, then $\Delta\hat{S}_k(h_m) > \Delta\hat{S}_k(h_j) \Rightarrow \Delta^2 S_k > 0$. Using (41) we obtain:

$$\Delta^2 S_k = \pm\epsilon S_k + (1 - p_k^+(h_j))[\overline{S}^{-}_{k+1}(h_m) - \overline{S}^{-}_{k+1}(h_j)] > 0$$

Since $\epsilon \ll 1$, therefore, for the above equation to be true we should have $\overline{S}^{-}_{k+1}(h_j) < \overline{S}^{-}_{k+1}(h_m)$. Thus, using corollary 1.2 we have $\overline{R}^{-}_{k+1}(h_j) > \overline{R}^{-}_{k+1}(h_m)$. ■

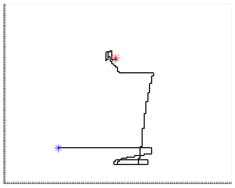
Remark 5.1: We haven't considered 2 cases in the above comparison of the algorithms, the first case is when $p_k^+(h_j) = p_k^+(h_m) + a$ such that one of $p_k^+(h_j)$ and $p_k^+(h_m)$ is greater than 0.5. This case is non-significant because speaking heuristically the value of $p_k^+(h_i)$; $\forall i$ is well below 0.5 even after the algorithm converges to the source location. The second case is when $p_k^+(h_j) > p_k^+(h_m)$ such that $\overline{R}^{-}_{k+1}(h_j) < \overline{R}^{-}_{k+1}(h_m)$. To evaluate the performance of the two algorithms under this condition we perform simulation.

VI. Simulations and Experimental Results

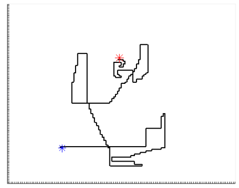
A. Simulation results

A number of simulation trials were performed to compare the expected rate and infotaxis algorithms. In each simulation a spherical robot of radius 0.1m was deployed to localize a source of plume in a 2D search space where wind is blowing in the negative y-axis direction at velocity, $V = 1$m/s. The plume source emits plume particles at rate $R_s = 1$ such that the plume particles have a life time of 2500 seconds and have a diffusivity of $D = 1$. The robot stops at every grid point in its path for $\Delta t = 5$sec. In each figure the source is denoted by a red asterisk (at top) whereas the the robot's starting location is denoted by a green asterisk (at bottom). Fig. 1 shows a comparison of path followed by the robot when using expected rate and infotaxis algorithms. While comparing the path for different grid sizes, it was found that when the grid size is small as compared to the robot size both the algorithms take comparable number of steps needed to localize the source. But, as the size of grid increases, the expected rate algorithm performs consistently better than infotaxis algorithm.

To verify this, simulations were performed for various grid

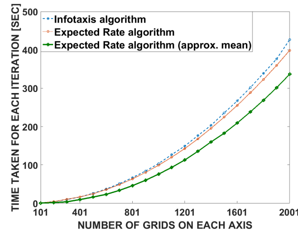


(a) Expected rate Algorithm

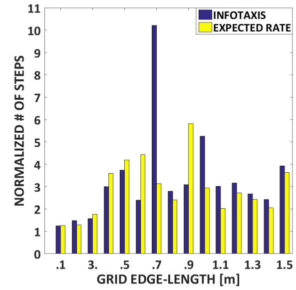


(b) Infotaxis algorithm

Fig. 1: Comparison for grid size 0.1m



(a) Time taken per interation



(b) # of steps (Normalized) taken for source localization

Fig. 2: Comparison graphs

sizes starting from 0.01m to 0.15m. Fig. 2b shows a plot of comparison of normalized number of steps for various step sizes. The number of steps taken by the robot for each algorithm is normalized with the minimum distance (in number of steps) between the source and initial position of the robot. We can see in the Fig. 2b that for grid size greater that 0.09m, expected rate algorithm consistently takes less number of steps than infotaxis algorithm.

Remark 6.1: A possible explanation for expected rate algorithm performing better than infotaxis algorithm, in case of large step size, can be their difference in the choice of next step when $p_k^+(h_j) > p_k^+(h_m)$, $\hat{R}(h_j) < \hat{R}(h_m)$ and $\Delta\hat{S}(h_j) < \Delta\hat{S}(h_m)$ for the next two possible steps h_j and h_m. Using simulation results, we can say that, it is better to choose the step which maximizes predicted rate of hits rather than the step which minimizes predicted change in entropy.

Fig. 2a shows a comparison of the time taken by one iteration of each algorithm for a given total number of grids. We can see that as the number of grids increase the difference between the time taken by each iteration becomes larger with infotaxis taking more time than expected rate algorithm.

B. Experimental results

A light source was used to represent a plume source and the intensity of the light was used to represent the rate of hits of plume particles encountered by a search robot.
We used a 60W bulb as light source and a Khepera robot which sensed ambient light intensity to calculate the number of hits. A lab space of 232cm$\times$175cm was used as the search space and the light source was placed at the right-center position. The search space was discretized into uniform grids and the robot was initialized at a position far from the source. Shaft encoders of the Khepera robot were used for localization of the robot. Fig. 3 shows the the snap shots of the Khepera successfully localizing the light source using the expected rate algorithm.

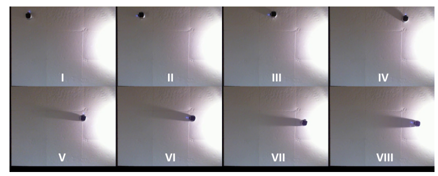


Fig. 3: Snapshots of the experiment

VII. Conclusion and Future work

We presented a novel framework of algorithms for source seeking and demonstrated that infotaxis algorithm fall under the framework. We introduced an algorithm based on the same framework which requires less amount of computation and produces identical control inputs under certain conditions. In future we would like to expand the algorithm to the case of multi-robot system with limited communication.

References

[1] N. Atanasov, J. Le Ny, N. Michael, and G. J. Pappas, "Stochastic source seeking in complex environments," in *Proc. 2012 IEEE International Conference on Robotics and Automation (ICRA2012)*. Minnesota, USA: IEEE, 2012, pp. 3013–3018.

[2] M. Guay and D. Dochain, "A time-varying extremum-seeking control approach," *Automatica, the IFAC Journal*, vol. 51, pp. 356–363, 2015.

[3] C. Zhang, D. Arnold, N. Ghods, A. Siranosian, and M. Krstic, "Source seeking with non-holonomic unicycle without position measurement and with tuning of forward velocity," *Systems & control letters*, vol. 56, no. 3, pp. 245–252, 2007.

[4] Y. Tian, W. Li, and F. Zhang, "Moth-inspired plume tracing via autonomous underwaer vehicle with only a pair of separated chemical sensors," in *Proc. MTS/IEEE OCEANS'15*. Washington D.C., USA: IEEE, 2015, pp. 1–8.

[5] S. Michalowsky and C. Ebenbauer, "Model-based extremum seeking for a class of nonlinear systems," in *Proc. 2015 American Control Conference (ACC2015)*. Boston, MA: IEEE, 2015, pp. 2026–2031.

[6] L. Brinón-Arranz, A. Seuret, and C. Canudas-de Wit, "Collaborative estimation of gradient direction by a formation of AUVs under communication constraints," in *Proc. 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC2011)*.

[7] W. Wu, D. Chang, and F. Zhang, "A bio-inspired robust 3d plume tracking strategy using mobile sensor networks," in *Proc. 52nd IEEE Conference on Decision and Control (CDC2013)*. Florence, Italy: IEEE, 2013, pp. 4571–4578.

[8] W. Wu and F. Zhang, "A speeding-up and slowing-down strategy for distributed source seeking with robustness analysis," *IEEE Transactions on Control of Networked Systems*, p. in print, 2015.

[9] M. Vergassola, E. Villermaux, and B. I. Shraiman, "infotaxis as a strategy for searching without gradients," *Nature*, vol. 445, no. 7126, pp. 406–409, 2007.

[10] Z. Siqi, C. Rongxin, and X. Demin, "Effectiveness of infotaxis algorithm for searching in dilute conditions," in *Proc. 31st Chinese Control Conference (CCC2012)*. Hefei, China: IEEE, 2012, pp. 5048–5053.

[11] M. Vergassola, E. Villermaux, and B. I. Shraiman, "Supplementary materials for infotaxis: as a strategy for searching without gradients," *Nature*, vol. 445, no. 7126, pp. 406–409, 2007.

[12] A. P. Dempster, N. M. Laird, and D. B. Rubin, "Maximum likelihood from incomplete data via the EM algorithm," *Journal of the royal statistical society. Series B (methodological)*, pp. 1–38, 1977.