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Adaptive Planar Curve Tracking Control with Unknown Curvature

Michael Malisoff   Fumin Zhang

***Abstract*— We provide a new adaptive controller for curve tracking in the plane under unknown curvature. We prove a global asymptotic stability theorem that ensures tracking of the curve and convergence of the curvature estimate to the unknown curvature. Our curvature identification result is an improvement over the known adaptive curve tracking results, which do not ensure parameter identification, or which identify the control gain without identifying curvatures.**

***Key Words*: curve tracking, robotics, robustness**

I. Introduction

This paper continues our search (begun in [8], [9], and [10]) for curve tracking methods that ensure stability under uncertainties. As noted in [7] and [9], curve tracking is important for navigating mobile robots; see, e.g., [11] for feedback controls for wheeled mobile robots which track obstacle boundaries, and [1], [2], and [3] for generalized adaptive robot controllers for under-actuated autonomous ships and other cases. For three dimensional cases and cooperative control for ocean sensing, see [4], [12], and [14].

Our work [8] proved robustness of the two dimensional curve tracking controls from [15] under polygonal state constraints; see Section III. This provided theoretical support for experimental evidence of robustness of curve tracking, which had been observed in experiments in ocean sampling [14], farming [6], obstacle avoidance in corridors [16], and ship control [3]. A key method in [8] involved robust forward invariance of closed sets $H \subseteq \mathbb{R}^2$, where the goal was to find largest constants $\delta_H > 0$ such that all trajectories starting in H, for all uncertainties that are bounded by δ_H, remain in H at all future times. This gave predictable tolerance and safety bounds by viewing the planar workspace as a nested union of forward invariant regions $\{H_i\}$, and proving input-to-state stability (or ISS) of the dynamics on each set H_i.

Our curve tracking research was motivated by our deployment of marine robots that searched for oil pollution from the 2010 Deepwater Horizon oil spill disaster [13]. Robust forward invariance can help ensure that curve tracking dynamics respect constraints in marine robotic implementations, such as no-collision constraints. Our experimental work tested our curve tracking controls under different control gains, and [9] extended [8] by proving ISS, robust forward invariance, and adaptive tracking under unknown control gains, leading to a new adaptive control analysis that identified control gains. However, [8]-[10] assume that the curvatures are known.

Therefore, we present a new adaptive controller for tracking curves. The significance of the present work is (a) our new dynamical extension that identifies unknown curvatures, (b) our proof of global asymptotic stability of the augmented tracking and curvature identification dynamics, and (c) our robust forward invariance approach to ensuring that the adaptively controlled dynamics respect certain state constraints. While our state constraints are hexagons like in [8] and [9], the analysis from [9] does not apply under unknown curvatures. Although [1] covers more complex models than ours (without identifying unknown model parameters), we believe that by identifying curvatures and proving robust forward invariance, our new work is valuable theory with the potential for marine applications with unknown curvatures.

II. Definitions and Notation

We use the standard classes of comparison functions $\mathcal{K}_\infty$ and $\mathcal{KL}$ from [5, Chapter 4]. Take any subset $\mathcal{G}$ of a Euclidean space and any point $\mathcal{E} \in \mathcal{G}$. We use the usual definitions of positive and negative definiteness with respect to $\mathcal{E}$, and moduli and nonstrict and strict Lyapunov functions with respect to $(\mathcal{E}, \mathcal{G})$ [8]. Let $|p|_\mathcal{E} = |p - \mathcal{E}|$ be the distance between any $p \in \mathcal{G}$ and $\mathcal{E}$, in the usual Euclidean metric.

Let $\mathcal{U}$ be any subset of a Euclidean space such that $0 \in \mathcal{U}$. Let $|f|_\mathcal{S}$ denote the essential supremum of any function f over any set $\mathcal{S}$, and $|f|_\infty$ denote its essential supremum over its entire domain. Take any forward complete system

$$\dot{x} = \mathcal{F}(x, \delta) \tag{1}$$

with state space $\mathcal{G}$ and measurable essentially bounded disturbances $\delta : [0, \infty) \to \mathcal{U}$, where $\mathcal{F}(\mathcal{E}, 0) = 0$. Let $\mathcal{S} \subseteq \mathcal{G}$ be a neighborhood of $\mathcal{E}$. The system is called *input-to-state stable (ISS)* with respect to $(\mathcal{U}, \mathcal{E}, \mathcal{S})$ provided that there are functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ and a modulus Λ with respect to $(\mathcal{E}, \mathcal{S})$ such that

$$|x(t, x_0, \delta)|_\mathcal{E} \leq \beta(\Lambda(x_0), t) + \gamma(|\delta|_{[0,t]}) \tag{2}$$

holds for all $t \geq 0$ and all solutions $x(t, x_0, \delta)$ of (1) corresponding to all initial states $x_0 \in \mathcal{S}$ and $\mathcal{U}$-valued δ's. This agrees with the usual ISS condition when $\mathcal{G} = \mathcal{S} = \mathbb{R}^n$, $\mathcal{E} = 0$, and $\Lambda(x) = |x|$. The special case where $\mathcal{F}$ only depends on x and $\gamma(|\delta|_{[0,t]})$ in (2) is not present is *global asymptotic stable (GAS)* with respect to $(\mathcal{E}, \mathcal{S})$. A set $\mathcal{H} \subseteq \mathcal{G}$ is *robustly forwardly invariant for (1) with disturbances valued in* $\mathcal{U}$ provided all trajectories of (1), with initial states in $\mathcal{H}$ and disturbances δ valued in $\mathcal{U}$, remain in $\mathcal{H}$ for all $t \geq 0$, i.e., $x(t, \mathcal{H}, \delta) \subseteq \mathcal{H}$ for all $t \geq 0$ and $\mathcal{U}$-valued δ's.

Malisoff is with the Dept. of Mathematics, 303 Lockett Hall, Louisiana State University, Baton Rouge, LA 70803-4918. malisoff@lsu.edu.
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Zhang is with the Georgia Tech School of Electrical and Computer Engineering, 777 Atlantic Dr. NW, Atlanta, GA 30332-0250, fumin@gatech.edu.
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III. Review of Model and Nonadaptive Cases

We review the curve tracking model and the relevant results from [8], which are needed for later sections. As noted in [8], the curve tracking dynamics can be simplified to

$$\dot{\rho} = -\sin(\phi), \quad \dot{\phi} = \tfrac{\kappa\cos(\phi)}{1+\kappa\rho} - u_0 + \Delta \tag{3}$$

where ρ is the distance between the robot and the curve being tracked, ϕ is the bearing, κ is the curvature at the closest point on the curve, u_0 is the steering control, the real valued essentially bounded function Δ represents uncertainty, and the state space is $\mathcal{X} = (0,\infty)\times(-\pi/2,\pi/2)$.

The work [15] designed a control to achieve asymptotic stabilization of an equilibrium $(\rho,\phi) = (\rho_0, 0)$ corresponding to a constant distance and zero bearing, which occurs when the robot moves parallel to the curve. Since κ in [8] and [15] was assumed to be known, they used a control of the form

$$u_0 = \tfrac{\kappa\cos(\phi)}{1+\kappa\rho} - h'(\rho)\cos(\phi) + \mu\sin(\phi) \tag{4}$$

where $\mu > 0$ is a steering constant. In [8], we assumed:

Assumption 1: The function $h : (0,\infty) \to [0,\infty)$ is C^2, h' has only finitely many zeros, $\lim_{\rho\to 0^+} h(\rho) = \lim_{\rho\to\infty} h(\rho) = \infty$, and there is a constant $\rho_0 > 0$ such that $h(\rho_0) = 0$. Also,

(a) There is a nondecreasing C^1 function $\gamma : [0,\infty) \to [\mu,\infty)$ such that $\gamma(h(\rho)) \geq 1+0.5\mu^2+h''(\rho)$ holds for all $\rho > 0$.

(b) There is a function $\Gamma \in \mathcal{K}_\infty \cap C^1$ such that $\Gamma(h(\rho)) \geq [h'(\rho)]^2$ for all $\rho > 0$.

(c) $h'(\rho)(\rho - \rho_0)$ is positive for all $\rho > 0$ except $\rho = \rho_0$, and $h''(\rho_0) > 0$. □

For instance, [8] shows that Assumption 1 holds for

$$\begin{aligned} h(\rho) &= \alpha\left(\rho + \tfrac{\rho_0^2}{\rho} - 2\rho_0\right), \\ \gamma(q) &= \tfrac{2}{\alpha^2\rho_0^4}(q+2\alpha\rho_0)^3 + 1 + 0.5\mu^2 + \mu \ \text{ and} \\ \Gamma(q) &= \tfrac{18\alpha}{\rho_0} q + \left(\tfrac{2}{\rho_0}\right)^4\left(\tfrac{9}{\alpha^2}\right) q^4 \end{aligned} \tag{5}$$

for any constants $\alpha > 0$ and $\rho_0 > 0$. Also, for any constant $L > 0$, [8] shows that the dynamics

$$\dot{\rho} = -\sin(\phi), \quad \dot{\phi} = h'(\rho)\cos(\phi) - \mu\sin(\phi), \tag{6}$$

obtained by substituting (4) into (3) and setting $\Delta = 0$, admits the strict Lyapunov function

$$\begin{aligned} U(\rho,\phi) \ = \ & -h'(\rho)\sin(\phi) + \tfrac{1}{\mu}\int_0^{V(\rho,\phi)} \gamma(m)\mathrm{d}m \\ & + L\Gamma(V(\rho,\phi)) + \tfrac{1}{2L}V(\rho,\phi) \end{aligned} \tag{7}$$

with respect to $((\rho_0,0),\mathcal{X})$, where γ and Γ are from Assumption 1 and V is the nonstrict Lyapunov function

$$V(\rho,\phi) = -\ln\big(\cos(\phi)\big) + h(\rho) \tag{8}$$

for (6) with respect to $((\rho_0,0),\mathcal{X})$. The function (8) was used in [15] with LaSalle Invariance to prove global asymptotic stability of $(\rho_0, 0)$. In fact, $\dot{V} = -\mu\sin^2(\phi)/\cos(\phi)$ holds along all solutions of (6) on our state space $\mathcal{X}$, which is a nonstrict Lyapunov function decay condition since $\dot{V}$ is zero at certain points outside the equilibrium $(\rho_0, 0)$.

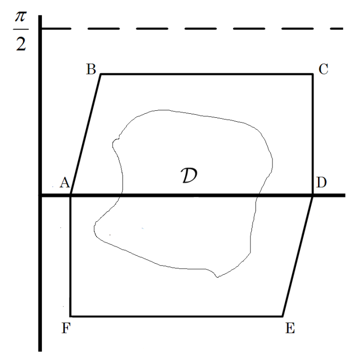

Fig. 1. Hexagon $H(\rho_*,\mu,K)$

Our work [8] proves that along all trajectories of (6) in $\mathcal{X}$, the function (7) satisfies $U \geq V$ and

$$\dot{U} \ \leq \ -0.5[h'(\rho)\cos(\phi)]^2 - \sin^2(\phi)\ , \tag{9}$$

so the right side of (9) is negative definite with respect to $(\rho_0, 0)$ on $\mathcal{X}$. The strict Lyapunov function decay condition (9) allowed us to prove ISS of the perturbed dynamics

$$\dot{\rho} = -\sin(\phi),\ \dot{\phi} = h'(\rho)\cos(\phi) - \mu\sin(\phi) + \Delta, \tag{10}$$

under suitable restrictions on $|\Delta|_\infty$, assuming that the curvature κ is a known positive constant.

The work [8] also gives this method for building robustly forwardly invariant sets. Given any constants $\rho_* \in (0, \rho_0/2)$ and $K > 1$, let $\mu \in (0, \pi/(2\rho_*))$ be any constant such that

$$\mu\tan(\mu\rho_*) > \max\left\{|h'(\rho)| : \rho_* \leq \rho \leq \rho_* + K\rho_0\right\}, \tag{11}$$

and $H(\rho_*,\mu,K) \subseteq \mathbb{R}^2$ be the closed region that is bounded by the hexagon that has the vertices $A = (\rho_*, 0)^\top$, $B = (2\rho_*, \mu\rho_*)^\top$, $C = (\rho_* + K\rho_0, \mu\rho_*)^\top$, $D = (\rho_* + K\rho_0, 0)^\top$, $E = (K\rho_0, -\mu\rho_*)^\top$, and $F = (\rho_*, -\mu\rho_*)^\top$. Then for each compact set $\mathcal{D} \subseteq \mathcal{X}$, we can choose ρ_*, μ, and K such that $\mathcal{D} \subseteq H(\rho_*,\mu,K)$; see Fig. 1. Set $\Delta_* = \min\{|h'(\rho)\cos(\phi)| : (\rho,\phi)^\top \in \mathrm{AB}\cup\mathrm{ED}\}$ and $\Delta_{**} = \min\{|h'(\rho)\cos(\phi) - \mu\sin(\phi)| : (\rho,\phi)^\top \in \mathrm{BC}\cup\mathrm{EF}\}$. Then

$$\min\{\Delta_*, \Delta_{**}\} > 0, \tag{12}$$

by (11) and Assumption 1, and [8] and [9] prove:

Lemma 1: Let Assumption 1 hold, $\rho_* \in (0, \rho_0/2)$ and $K > 1$ and $L > 0$ be any constants, and $\mu \in (0, \pi/(2\rho_*))$ be any constant satisfying (11). Then the following hold: (a) For any constant $\bar{\Delta} \in (0, \min\{\Delta_*, \Delta_{**}\})$, the set $H(\rho_*,\mu,K)$ is robustly forwardly invariant for (10) with disturbances valued in $\mathcal{U} = [-\bar{\Delta}, \bar{\Delta}]$. (b) For each constant $\tilde{\Delta} > \min\{\Delta_*, \Delta_{**}\}$, we can find a point $\tilde{p}$ on the boundary of $H(\rho_*,\mu,K)$ such that the trajectory of (10) starting at $\tilde{p}$ for one of the constant perturbations $\Delta = \pm\tilde{\Delta}$ exits the hexagon. (c) There is a constant $v_0 > 0$ such that the function (7) satisfies $U(\rho,\phi) \geq v_0|(\rho - \rho_0, \phi)|^2$ for all $(\rho,\phi) \in H(\rho_*,\mu,K)$. □

The preceding lemma implies that $\min\{\Delta_*, \Delta_{**}\}$ is the maximal allowable perturbation bound for maintaining robust forward invariance of $H(\rho_*,\mu,K)$. We also use the following, whose proof consists of Step 3 of [9, Appendix B], and differs from a standard sufficient condition for ISS because its Lyapunov decay condition is of integral ISS type:

Lemma 2: Let $\mathcal{X}^\sharp$ be a bounded robustly forwardly invariant set for some dynamics of the form (1) with disturbances $\delta : [0,\infty) \to [-\delta_*, \delta_*]$ that are bounded by a constant $\delta_* > 0$, where $\mathcal{F}(0,0) = 0$. Let $V^\sharp : \mathcal{O} \to [0,\infty)$ be C^1 on some open set $\mathcal{O}$ containing $\mathcal{X}^\sharp$ and admit a constant $\underline{v} > 0$, a continuous positive definite function $\alpha_0 : [0,\infty) \to [0,\infty)$, a function $\bar{\gamma} \in \mathcal{K}_\infty$, and a modulus Λ with respect to $(0, \mathcal{X}^\sharp)$ such that $\dot{V}^\sharp \le -\alpha_0(V^\sharp) + \bar{\gamma}(|\delta|)$ and $\underline{v}|x|^2 \le V^\sharp(x) \le \Lambda(x)$ hold along all trajectories of (1) starting in $\mathcal{X}^\sharp$ for all measurable disturbances $\delta : [0,\infty) \to [-\delta_*, \delta_*]$. Then, we can construct functions $\beta^\sharp \in \mathcal{KL}$ and $\gamma^\sharp \in \mathcal{K}_\infty$ such that $|x(t)| \le \beta^\sharp(\Lambda(x(0)), t) + \gamma^\sharp(|\delta|_{[0,t]})$ holds along all trajectories of (1) starting in $\mathcal{X}^\sharp$ for all choices of δ, so (1) is ISS with respect to $([-\delta_*, \delta_*], 0, \mathcal{X}^\sharp)$. □

IV. Main Adaptive Control and Tracking Result

A. Statement of Result

We now leverage the results from the preceding section, to study the two dimensional curve tracking dynamics

$$\dot{\rho} = -\sin\phi, \quad \dot{\phi} = \frac{\kappa\cos\phi}{1+\kappa\rho} - u_2 \tag{13}$$

with unknown constant curvatures κ (but see Section V for extensions with disturbances and nonconstant curvatures). The control u_2 in (13) will differ from the control u_0 from (4), since we will no longer assume that the curvature κ is available for use in the control. This produces completely different adaptive control problems from the ones in [9] and [10], which covered known curvatures. By writing

$$\frac{\kappa}{1+\kappa\rho} = \frac{\frac{\kappa}{1+\kappa\rho_0}}{1+\frac{\kappa}{1+\kappa\rho_0}(\rho-\rho_0)} = \frac{\kappa_0}{1+\kappa_0(\rho-\rho_0)} \tag{14}$$

for our desired constant distance $\rho_0 > 0$, where $\kappa_0 = \kappa/(1+\kappa\rho_0)$, we will rescale κ to replace $\kappa/(1+\kappa\rho)$ by $\kappa_0/(1+\kappa_0(\rho-\rho_0))$ in (13) and so also in what follows; see Section V for motivation for the transformation (14).

Even though κ_0 is unknown, we assume that we know constants $\underline{c}$ and $\bar{c}$ such that $\kappa_0 \in (\underline{c}, \bar{c})$, and that $\kappa_0 > 0$. We also assume that Assumption 1 holds. Using the strict Lyapunov function (7), we use the estimator

$$\dot{\hat{\kappa}}_0 = (\hat{\kappa}_0 - \underline{c})(\bar{c} - \hat{\kappa}_0)\frac{\cos(\phi)}{(1+(\rho-\rho_0)\hat{\kappa}_0)^2}\frac{\partial U}{\partial \phi}(\rho,\phi) \tag{15}$$

for the unknown scaled curvature κ_0. This is valid, because our choice of U in (7) does not depend on κ. Later we specify our state spaces in such a way that $1 + (\rho-\rho_0)\hat{\kappa}_0$ and $1 + (\rho - \rho_0)\kappa_0$ stay positive. We also use the controller

$$u_2 = \frac{\hat{\kappa}_0\cos(\phi)}{1+\hat{\kappa}_0(\rho-\rho_0)} - h'(\rho)\cos(\phi) + \mu\sin(\phi) \tag{16}$$

where h satisfies Assumption 1, so $h''(\rho_0) > 0$.

Then, after applying (14) to (13) and taking a common denominator, we conclude that the closed loop dynamics for $(\tilde{q}, \tilde{\kappa}_0) = (\tilde{q}_1, \tilde{q}_2, \tilde{\kappa}_0) = (\rho - \rho_0, \phi, \hat{\kappa}_0 - \kappa_0)$ are

$$\left\{ \begin{array}{rcl} \dot{\tilde{q}}_1 &=& -\sin(\tilde{q}_2) \\ \dot{\tilde{q}}_2 &=& h'(\tilde{q}_1 + \rho_0)\cos(\tilde{q}_2) - \mu\sin(\tilde{q}_2) \\ && - \frac{\tilde{\kappa}_0\cos(\phi)}{(1+\kappa_0(\rho-\rho_0))(1+(\kappa_0+\tilde{\kappa}_0)(\rho-\rho_0))} \\ \dot{\tilde{\kappa}}_0 &=& \frac{(\kappa_0+\tilde{\kappa}_0-\underline{c})(\bar{c}-\kappa_0-\tilde{\kappa}_0)}{(1+(\rho-\rho_0)\hat{\kappa}_0)^2}\cos(\tilde{q}_2)\frac{\partial U}{\partial\phi}(\rho,\phi). \end{array} \right. \tag{17}$$

To specify the state constraint set for (17), fix any one of the compact robustly forwardly invariant sets $\mathcal{S} = H(\rho_*, \mu, K)$ from the previous section for the dynamics (10), and any constant $\bar{\Delta} \in (0, \min\{\Delta_*, \Delta_{**}\})$, where $\min\{\Delta_*, \Delta_{**}\}$ is the perturbation bound from Lemma 1 for $\mathcal{S}$.

Finally, choosing any functions γ and Γ that satisfy Assumption 1 and any constant $L > 0$, and defining V by (8), we fix any positive constants $\bar{\mathcal{M}}_1$ and $\bar{\mathcal{M}}_2$ such that

$$\begin{array}{l} \bar{\mathcal{M}}_1 \ge \frac{1}{\mu}\gamma(V(\rho,\phi)) + L\Gamma'(V(\rho,\phi)) + \frac{1}{2L} \;\text{ and} \\ \bar{\mathcal{M}}_2 \ge \frac{\rho-\rho_0}{h'(\rho)}\max\left\{1, \frac{\rho-\rho_0}{h'(\rho)\cos^2(\phi)}\right\} \end{array} \tag{18}$$

hold for all $(\rho,\phi) \in \mathcal{S}$ such that $\rho \ne \rho_0$. These constants exist by the continuity of V on the compact set $\mathcal{S}$, the continuity of γ and Γ', and the facts that $\mu\rho_* < \pi/2$ and $h''(\rho_0) > 0$, combined with L'Hopital's Rule to bound $(\rho-\rho_0)/h'(\rho)$. Our main result is:

Theorem 1: Let $\mathcal{S}$, $\bar{\Delta}$, ρ_0, h, U, $\bar{\mathcal{M}}_1$, and $\bar{\mathcal{M}}_2$ satisfy the above requirements and the constants $\underline{c} \ge 0$ and $\bar{c} > \underline{c}$ satisfy

$$\bar{c} < \underline{c} + \min\left\{\frac{\bar{\Delta}}{4}, \frac{1}{2\sqrt{\bar{\mathcal{M}}_1}}, \frac{1}{2\sqrt{2\bar{\mathcal{M}}_2(2+\bar{\mathcal{M}}_1)}}\right\} \tag{19}$$

$$\text{and } \; \bar{c} < \frac{1}{2(\rho_0-\rho_*)}. \tag{20}$$

Then, (17) is GAS with respect to $(0, \mathcal{S}^\sharp)$ where $\mathcal{S}^\sharp = \{(\tilde{q}, \tilde{\kappa}_0) : \tilde{q} + (\rho_0, 0) \in \mathcal{S}, \tilde{\kappa}_0 + \kappa_0 \in (\underline{c}, \bar{c})\}$. □

B. Key Forward Invariance Lemma

To prove Theorem 1, we first prove:

Lemma 3: Let the assumptions of Theorem 1 hold. Then for each initial state $(\tilde{q}(0), \tilde{\kappa}_0(0)) \in \mathcal{S}^\sharp$ for (17), the solution $(\tilde{q}(t), \tilde{\kappa}_0(t)) = (\rho(t) - \rho_0, \phi(t), \hat{\kappa}_0(t) - \kappa_0)$ for (17) satisfies

$$\min\{1 + \kappa_0\tilde{q}_1(t), 1 + (\kappa_0 + \tilde{\kappa}_0(t))\tilde{q}_1(t)\} \ge \tfrac{1}{2} \tag{21}$$

for all $t \ge 0$ and $(\tilde{q}(t), \tilde{\kappa}(t)) \in \mathcal{S}^\sharp$ for all $t \ge 0$. □

Proof: First note that we can find a rectangle $[\rho_{\min}, \rho_{\max}] \times [\phi_{\min}, \phi_{\max}] \subseteq (0,\infty) \times (-\pi/2, \pi/2)$ such that $\mathcal{S} \subseteq (\rho_{\min}, \rho_{\max}) \times (\phi_{\min}, \phi_{\max})$ and such that

$$\bar{c} < \frac{1}{2(\rho_0 - \rho_{\min})} \tag{22}$$

holds. This follows by choosing $\rho_{\min} < \rho_*$ close enough to ρ_*, using the strictness of inequality (20).

We next check that for all $\rho \in [\rho_{\min}, \rho_{\max}]$ and $r \in [\underline{c}, \bar{c}]$, we have

$$1 + r(\rho - \rho_0) \ge \tfrac{1}{2}. \tag{23}$$

This will give (21), after we show that $(\tilde{q}(t), \tilde{\kappa}(t)) \in \mathcal{S}^\sharp$ for all $t \ge 0$ when $(\tilde{q}(0), \tilde{\kappa}(0)) \in \mathcal{S}^\sharp$. There are two cases. Case 1: If $\rho - \rho_0 \ge 0$, then $1 + r(\rho-\rho_0) \ge 1$, since $r \ge \underline{c} \ge 0$. Case 2: If $\rho - \rho_0 < 0$, then (22) gives $r(\rho - \rho_0) \ge r(\rho_{\min} - \rho_0) \ge \bar{c}(\rho_{\min} - \rho_0) > -1/2$, since $\rho_{\min} < \rho_0$. Hence, (23) holds.

Next, consider any initial state $(\tilde{q}(0), \tilde{\kappa}_0(0)) \in \mathcal{S}^\sharp$. The existence of the unique $\mathcal{S}^\sharp$-valued maximal solution of (17) starting at $(\tilde{q}(0), \tilde{\kappa}_0(0))$ on some maximal interval of the form $[0, t_{\max})$ follows from the local Lipschitzness of the right side of (17), since the denominators in (17) are positive at $t = 0$. Also, (19) gives $|\tilde{\kappa}_0(t)| \le \bar{c} - \underline{c} < \bar{\Delta}/4$, so (23) (with $r=\kappa_0$ and then $r=\hat{\kappa}_0$ and ρ depending on t) gives

$$\left|\frac{\tilde{\kappa}_0(t)\cos(\phi(t))}{(1+\kappa_0(\rho(t)-\rho_0))(1+(\kappa_0+\tilde{\kappa}_0(t))(\rho(t)-\rho_0))}\right| < \bar{\Delta} \tag{24}$$

for all times t at which $(\rho(t),\phi(t)) \in [\rho_{\min},\rho_{\max}] \times [\phi_{\min},\phi_{\max}]$ and $\hat{\kappa}_0(t) \in (\underline{c},\bar{c})$. Suppose that $(\tilde{q}(t),\tilde{\kappa}_0(t))$ did not stay in $\mathcal{S}^\sharp$, for the sake of obtaining a contradiction. In that case, there is a maximal time t_* such that $(\tilde{q}(r),\tilde{\kappa}_0(r)) \in \mathcal{S}^\sharp$ for all $r\in[0,t_*]$. Moreover, $(\rho(t_*),\phi(t_*))$ is at the boundary $\partial\mathcal{S}$ of $\mathcal{S}$, because a uniqueness of solutions argument (which is analogous to [9, Footnote 2]) ensures that $\tilde{\kappa}_0(t)$ cannot reach $-\kappa_0+\underline{c}$ or $-\kappa_0+\bar{c}$, so $\hat{\kappa}_0$ stays in $(\underline{c},\bar{c})$.

Since the maximal solution is defined on a half open interval of the form $[0,t_{\max})$ and $\mathcal{S} \subseteq (\rho_{\min},\rho_{\max}) \times (\phi_{\min},\phi_{\max})$ is compact, there is a constant $\varepsilon > 0$ such that the function $\psi : [0,\varepsilon] \to \mathbb{R}^3$ defined by $\psi(t) = (\psi_1(t),\psi_2(t),\psi_3(t)) = (\tilde{q}(t_*+t),\tilde{\kappa}_0(t_*+t))$ is such that $(\psi_1(t)+\rho_0,\psi_2(t))$ starts in $\partial\mathcal{S}$, takes all values in $[\rho_{\min},\rho_{\max}]\times[\phi_{\min},\phi_{\max}]$, and solves (10) on $[0,\varepsilon]$ with

$$\Delta(t) = -\frac{\tilde{\kappa}_0(t_*+t)\cos(\psi_2(t))}{(1+\kappa_0\psi_1(t))(1+(\kappa_0+\tilde{\kappa}_0(t_*+t))\psi_1(t))}$$

and $\max_{t\in[0,\varepsilon]}|\Delta(t)| < \bar{\Delta}$, by (24). This ε exists, by the continuity of $(\tilde{q},\tilde{\kappa}_0)$. Since $\bar{\Delta}$ is a perturbation bound for maintaining forward invariance of $\mathcal{S}$ and $\psi(t)$ starts in $\mathcal{S}^\sharp$, ψ stays in $\mathcal{S}^\sharp$ on $[0,\varepsilon]$. Hence, $(\tilde{q},\tilde{\kappa}_0)$ stays in $\mathcal{S}^\sharp$ on $[0,t_*+\varepsilon]$. This contradicts the maximality of t_*, so Lemma 3 holds. ■

C. Stability Analysis and Curvature Identification

Recall that along all trajectories of (10) in $\mathcal{S}$, the function U satisfies the strict decay condition (9), when $\Delta = 0$. Hence, since $\hat{\kappa}_0 = \tilde{\kappa}_0 + \kappa_0$ stays in $(\underline{c},\bar{c})$, the function

$$U^\sharp(\rho,\phi,\tilde{\kappa}_0) = U(\rho,\phi) + \int_0^{\tilde{\kappa}_0} \frac{\ell}{(\ell+\kappa_0-\underline{c})(\bar{c}-\ell-\kappa_0)}\mathrm{d}\ell \tag{25}$$

satisfies

$$\begin{aligned}\dot{U}^\sharp \;\le\;& -\tfrac{1}{2}(h'(\rho)\cos(\phi))^2 - \sin^2(\phi)\\ &+ \left\{\frac{\partial U}{\partial\phi}(\rho,\phi)\frac{\cos(\phi)(-\tilde{\kappa}_0^2)(\rho-\rho_0)}{(1+\hat{\kappa}_0(\rho-\rho_0))^2(1+\kappa_0(\rho-\rho_0))}\right\}.\end{aligned} \tag{26}$$

along all trajectories of (17) in its forwardly invariant set $\mathcal{S}^\sharp$. By (18) and the facts that $(\rho(t),\phi(t))$ stays in $\mathcal{S}$ and $\frac{\partial V}{\partial\phi}(\rho,\phi) = \tan(\phi)$ on $\mathcal{S}$, our choice (7) of U gives

$$\begin{aligned}\left|\frac{\partial U}{\partial\phi}(\rho,\phi)\right| \;\le\;& \Big|-h'(\rho)\cos(\phi) + \Big(\tfrac{1}{\mu}\gamma(V(\rho,\phi))\\ &+ L\Gamma'(V(\rho,\phi)) + \tfrac{1}{2L}\Big)\tan(\phi)\Big|\\ \le\;& \big|h'(\rho)\cos(\phi)\big| + \bar{\mathcal{M}}_1\big|\tan(\phi)\big|\end{aligned} \tag{27}$$

on our robustly forwardly invariant set $\mathcal{S}$.

Hence, we can use our choice of $\bar{\mathcal{M}}_2$, our lower bound (21), the relation $1/(.5)^3 = 8$, and the triangle inequality to upper bound the quantity in curly braces in (26) by

$$\begin{aligned}&8\tilde{\kappa}_0^2\big(|h'(\rho)(\rho-\rho_0)|\cos^2(\phi) + \bar{\mathcal{M}}_1|\sin(\phi)(\rho-\rho_0)|\big)\\ &\le 8\tilde{\kappa}_0^2\big(|h'(\rho)(\rho-\rho_0)|\cos^2(\phi) + \tfrac{\bar{\mathcal{M}}_1}{2}\sin^2(\phi)\\ &\quad + \tfrac{\bar{\mathcal{M}}_1}{2}(\rho-\rho_0)^2\big)\\ &\le 8\tilde{\kappa}_0^2\Big(\bar{\mathcal{M}}_2(h'(\rho)\cos(\phi))^2 + \tfrac{\bar{\mathcal{M}}_1}{2}\sin^2(\phi)\\ &\quad + \tfrac{\bar{\mathcal{M}}_1}{2}\bar{\mathcal{M}}_2(h'(\rho)\cos(\phi))^2\Big).\end{aligned} \tag{28}$$

Also, $|\tilde{\kappa}_0|$ is bounded by $\bar{c}-\underline{c}$, and (19) implies that

$$\begin{aligned}&8(\bar{c}-\underline{c})^2\bar{\mathcal{M}}_2(1+0.5\bar{\mathcal{M}}_1) < \tfrac{1}{2} \text{ and}\\ &8(\bar{c}-\underline{c})^2(\bar{\mathcal{M}}_1/2) < 1.\end{aligned} \tag{29}$$

Then (26) and (28) give a constant $\beta_0 > 0$ such that

$$\begin{aligned}\dot{U}^\sharp \;\le\;& -\tfrac{1}{2}(h'(\rho)\cos(\phi))^2 - \sin^2(\phi)\\ &+ 8(\bar{c}-\underline{c})^2\Big(\bar{\mathcal{M}}_2\Big(1+\tfrac{\bar{\mathcal{M}}_1}{2}\Big)(h'(\rho)\cos(\phi))^2\\ &+ \tfrac{\bar{\mathcal{M}}_1}{2}\sin^2(\phi)\Big)\\ \le\;& -\beta_0\big((h'(\rho)\cos(\phi))^2 + \sin^2(\phi)\big)\end{aligned} \tag{30}$$

along all trajectories of (17) in $\mathcal{S}^\sharp$.

We next convert $U^\sharp$ into a strict Lyapunov function $V^\sharp$ for (17) with respect to $(0,\mathcal{S}^\sharp)$, having the form

$$V^\sharp(\tilde{q},\tilde{\kappa}_0) = \bar{\mathcal{M}}_3 U^\sharp(\rho,\phi,\tilde{\kappa}_0) + \tilde{q}_2\tilde{\kappa}_0 \tag{31}$$

for a suitable constant $\bar{\mathcal{M}}_3 > 0$. We first pick a constant $\bar{\mathcal{G}}_1 > 0$ such that

$$\left|\dot{\tilde{\kappa}}_0\right| \le \bar{\mathcal{G}}_1|\tilde{q}| \;\text{ and }\; U^\sharp(\rho,\phi,\tilde{\kappa}_0) \ge \frac{|(\tilde{q},\tilde{\kappa}_0)|^2}{\bar{\mathcal{G}}_1} \tag{32}$$

hold on $\mathcal{S}^\sharp$. The constant $\bar{\mathcal{G}}_1$ exists because of part (c) of Lemma 1, combined with (21) and (27). Then we can use (21), (32), the fact that $h'(\rho_0) = 0$, and the bounds $|\tilde{\kappa}_0(t)| \le \bar{c}-\underline{c}$ and $\cos(\phi) \ge \cos(\mu\rho_*) > 0$ to find constants $\bar{\mathcal{G}}_2 > 0$ and $\bar{\mathcal{G}}_3 > 0$ such that on $\mathcal{S}^\sharp$, we have

$$\begin{aligned}\tfrac{d}{dt}(\tilde{q}_2\tilde{\kappa}_0) \;\le\;& \Big(h'(\tilde{q}_1+\rho_0)\cos(\tilde{q}_2) - \mu\sin(\tilde{q}_2)\\ &- \frac{\tilde{\kappa}_0\cos(\phi)}{(1+\kappa_0(\rho-\rho_0))(1+\hat{\kappa}_0(\rho-\rho_0))}\Big)\tilde{\kappa}_0\\ &+ \bar{\mathcal{G}}_1|\tilde{q}|^2\\ \le\;& \bar{\mathcal{G}}_2\left(|\tilde{\kappa}_0||\tilde{q}| + |\tilde{q}|^2\right) - \bar{\mathcal{G}}_3\tilde{\kappa}_0^2\\ \le\;& \bar{\mathcal{G}}_2\left(\tfrac{\bar{\mathcal{G}}_2}{2\bar{\mathcal{G}}_3}|\tilde{q}|^2 + \tfrac{\bar{\mathcal{G}}_3}{2\bar{\mathcal{G}}_2}\tilde{\kappa}_0^2 + |\tilde{q}|^2\right) - \bar{\mathcal{G}}_3\tilde{\kappa}_0^2\\ =\;& \bar{\mathcal{G}}_2\left(\tfrac{\bar{\mathcal{G}}_2}{2\bar{\mathcal{G}}_3}+1\right)|\tilde{q}|^2 - \tfrac{1}{2}\bar{\mathcal{G}}_3\tilde{\kappa}_0^2.\end{aligned} \tag{33}$$

We can also find a constant $c_0 > 0$ such that $\beta_0\big((h'(\rho)\cos(\phi))^2 + \sin^2(\phi)\big) \ge c_0|\tilde{q}|^2$, and so also $\dot{U}^\sharp \le -c_0|\tilde{q}|^2$ on $\mathcal{S}^\sharp$, using (30), L'Hopital's rule, and the facts that $h''(\rho_0) > 0$ and $\cos(\phi) \ge \cos(\mu\rho_*) > 0$. Hence, the choice

$$\bar{\mathcal{M}}_3 = 1 + \bar{\mathcal{G}}_1 + \tfrac{1}{c_0}\bar{\mathcal{G}}_2\left(\tfrac{\bar{\mathcal{G}}_2}{2\bar{\mathcal{G}}_3}+1\right) \tag{34}$$

implies that on $\mathcal{S}^\sharp$, we have

$$\begin{aligned}\dot{V}^\sharp(\tilde{q},\tilde{\kappa}_0) \;\le\;& -\bar{\mathcal{M}}_3 c_0|\tilde{q}|^2 + \tfrac{d}{dt}(\tilde{q}_2\tilde{\kappa}_0)\\ \le\;& -c_0|\tilde{q}|^2 - \bar{\mathcal{G}}_2\left(\tfrac{\bar{\mathcal{G}}_2}{2\bar{\mathcal{G}}_3}+1\right)|\tilde{q}|^2 + \tfrac{d}{dt}(\tilde{q}_2\tilde{\kappa}_0)\\ \le\;& -\underline{v}|(\tilde{q},\tilde{\kappa}_0)|^2 \;\text{ and}\\ V^\sharp(\tilde{q},\tilde{\kappa}_0) \;\ge\;& \bar{\mathcal{M}}_3|(\tilde{q},\tilde{\kappa}_0)|^2/\bar{\mathcal{G}}_1 - 0.5|\tilde{q}|^2 - 0.5\tilde{\kappa}_0^2\\ \ge\;& \underline{v}|(\tilde{q},\tilde{\kappa}_0)|^2,\end{aligned} \tag{35}$$

where $\underline{v} = \min\{c_0, \bar{\mathcal{G}}_3/2, 1/2\}$, by (31)-(33).

Also, the following variant of an argument from Appendix B in [9] provides a positive definite function α_0 such that

$$\alpha_0\big(V^\sharp(\tilde{q},\tilde{\kappa}_0)\big) \le \underline{v}|(\tilde{q},\tilde{\kappa}_0)|^2 \text{ for all } (\tilde{q},\tilde{\kappa}_0) \in \mathcal{S}^\sharp. \tag{36}$$

First choose any constant $\varepsilon \in (0, 0.5\min\{\bar{c}-\kappa_0, \kappa_0-\underline{c}\})$.

Then (i) there is a function $\alpha_1 \in \mathcal{K}_\infty$ such that $V^\sharp(\tilde{q}, \tilde{\kappa}_0) \leq \alpha_1(|(\tilde{q}, \tilde{\kappa}_0)|)$ for all $(\tilde{q}, \tilde{\kappa}_0) \in \mathcal{S}^\sharp$ such that $\tilde{\kappa}_0 \in [\underline{c} - \kappa_0 + \varepsilon, \bar{c} - \kappa_0 - \varepsilon]$ and (ii) there is a constant $c_1 > 0$ such that $c_1 \leq \underline{v}|(\tilde{q}, \tilde{\kappa}_0)|^2$ for all other points $(\tilde{q}, \tilde{\kappa}_0) \in \mathcal{S}^\sharp$. Hence, by separately considering the cases (i) and (ii), we conclude that (36) holds with $\alpha_0(r) = \min\{c_1, \underline{v}[\alpha_1^{-1}(r)]^2\}$. This gives

$$\dot{V}^\sharp(\tilde{q}, \tilde{\kappa}_0) \leq -\alpha_0(V^\sharp(\tilde{q}, \tilde{\kappa}_0)) \tag{37}$$

along all trajectories of the (17) in $\mathcal{S}^\sharp$. Hence, Theorem 1 follows from (35), (37), and Lemma 2 with $\delta = 0$.

V. Discussion on Assumptions and Extensions

Theorem 1 applies when κ_0 in (17) lies in $(\underline{c}, \bar{c})$, for any constants $\underline{c} \geq 0$ and $\bar{c} > \underline{c}$ satisfying (19)-(20). However, our derivation of (17) was based on rescaling the curvature. The rescaling was used to introduce the $\rho - \rho_0$ terms in (28). In terms of the curvature parameter κ from the original model (13), our bound requirements are $\underline{c} < \kappa/(1 + \kappa\rho_0) < \bar{c}$.

In robotics, curve tracking is usually done for straight lines or circles, or for curves whose curvatures change slowly relative to the convergence speed of the robot and so can be regarded as constant. However, we can generalize our results to allow nonconstant curvatures, as follows. Assume that the unknown curvature is some function $\kappa^\sharp(s) = \kappa + \eta(s)$ of the curve length s for some constant $\kappa > 0$, and that we know a constant $\bar{\delta} \in (0, \kappa)$ such that $\sup_s |\eta(s)| \leq \bar{\delta}$. Then replacing κ by $\kappa^\sharp(s)$ in (13) produces

$$\dot{\rho} = -\sin\phi, \quad \dot{\phi} = \tfrac{\kappa\cos\phi}{1+\kappa\rho} - u_2 + \delta \tag{38}$$

with the unknown constant nominal curvature $\kappa \geq 0$, where

$$\delta = \tfrac{\cos(\phi)\eta(s)}{(1+(\kappa+\eta(s))\rho)(1+\kappa\rho)} \tag{39}$$

is bounded by $\bar{\delta}$. Then we can also prove ISS properties.

To prove this generalization, we replace the bound $\bar{c} < \underline{c} + \bar{\Delta}/4$ from (19) by $\bar{c} < \underline{c} + (\bar{\Delta} - \bar{\delta})/4$ (which requires that $\bar{\delta} < \bar{\Delta}$), and then argue as before (except with Lemma 3 generalized to a robust forward invariance result for a perturbed version of (17)) to conclude that the perturbed augmented tracking and curvature identification dynamics satisfy ISS with respect to $([-\bar{\delta}, \bar{\delta}], 0, \mathcal{S}^\sharp)$ and the perturbation $\delta(t)$, where $\mathcal{S}^\sharp$ is from Theorem 1. This follows from the ISS conclusion of Lemma 2 and the strict decay condition on U.

Also, Theorem 1 remains true if we fix any constant $\lambda \in (0, 1)$ and replace (19)-(20) by

$$\bar{c} < \underline{c} + \min\left\{\lambda^2\bar{\Delta}, \sqrt{\tfrac{2\lambda^3}{\bar{\mathcal{M}}_1}}, \sqrt{\tfrac{\lambda^3}{\bar{\mathcal{M}}_2(2+\bar{\mathcal{M}}_1)}}\right\} \tag{40}$$

$$\text{and } \bar{c} < \tfrac{1-\lambda}{\rho_0 - \rho_*}, \tag{41}$$

by replacing the lower bound in (21) by λ, and replacing the 8's in (28)-(29) by λ^{-3}. Also, we can satisfy our requirements using $h(\rho) = \alpha(\rho - \rho_0)^2$ for any constant $\alpha > 0$, by only requiring the conditions from Assumption 1 for all $\rho \in [\rho_*, \rho_* + K\rho_0]$, since this still gives (9) on $\mathcal{S}$.

For example, if we take $h(\rho) = (\rho - \rho_0)^2$, $\rho_0 = 1$, $\rho_* = 0.25$, $K = 5/4$, $L = 0.4$, and $\mu \in (0, \pi)$ close enough to π, then the ρ values occurring for points in $\mathcal{S}$ are in $[0.25, 1.5]$, the ϕ values occurring in $\mathcal{S}$ are in $[-\pi/4, \pi/4]$, and we can choose the constant function $\gamma(\ell) = 7.9$, $\Gamma(\ell) = 4\ell$, $\bar{\mathcal{M}}_1 = 5.4$, $\bar{\mathcal{M}}_2 = 0.5$, $\Delta_* = 0.4$, $\Delta_{**} = 1.3$, and $\bar{\Delta} = 0.4$ to satisfy the requirements from Assumption 1 and the required lower bounds (18). Then, taking $\lambda = 7/8$ lets us satisfy (40)-(41) with $\underline{c} = 0$ and an upper bound $\bar{c} \approx 0.2$.

VI. Conclusions

Adaptive planar curve tracking under unknown curvatures is important for the control of marine robots. While our works [9] and [10] solve adaptive tracking and parameter identification problems under unknown control gains, here we solved a complementary problem, where the control gains are known but where our adaptive controller can identify unknown curvatures. Our strict Lyapunov function allows us to cover nonconstant curvatures, and our robust forward invariance approach lets us satisfy certain state constraints. In future work, we hope to apply our new curvature identification method in field work with marine robots.

References

[1] A. Aguiar and J. Hespanha. Trajectory-tracking and path-following of underactuated autonomous vehicles with parametric modeling uncertainty. *IEEE Trans. Automat. Control*, 52(8):1362-1379, 2007.

[2] E. Borhaug and K. Pettersen. Adaptive way-point tracking control for underactuated autonomous vehicles. In *Proc. 44th IEEE Conf. Decision and Control*, Seville, Spain, 2005, pp. 4028-4034.

[3] K. Do and J. Pan. Robust path-following of underactuated ships: Theory and experiments on a model ship. *Ocean Engineering*, 33(10):1354-1372, 2006.

[4] E. Justh and P. Krishnaprasad. Natural frames and interacting particles in three dimensions. In *Proc. 45th IEEE Conf. on Decision and Control and European Control Conf.*, San Diego, CA, 2005, pp. 2841-2846.

[5] H. Khalil. *Nonlinear Systems, Third Edition*. Prentice Hall, Englewood Cliffs, NJ, 2002.

[6] R. Lenain, B. Thuilot, C. Cariou, and P. Martinet. High accuracy path tracking for vehicles in presence of sliding: Application to farm vehicle automatic guidance for agricultural tasks. *Autonomous Robots*, 21(1):79-97, 2006.

[7] V. Lumelsky and A. Stepanov. Path planning strategies for a point mobile automaton moving amidst unknown obstacles of arbitrary shape. *Algorithmica*, 2(2):403-430, 1987.

[8] M. Malisoff, F. Mazenc, and F. Zhang. Stability and robustness analysis for curve tracking control using input-to-state stability. *IEEE Trans. Automat. Control*, 57(5):1320-1326, 2012.

[9] M. Malisoff and F. Zhang. Adaptive control for planar curve tracking under controller uncertainty. *Automatica*, 49(5):1411-1418, 2013.

[10] M. Malisoff and F. Zhang. Robustness of adaptive control under time delays for three-dimensional curve tracking. *SIAM Journal on Control and Optimization*, 53(4):2203-2236, 2015.

[11] A. Micaelli and C. Samson. Trajectory tracking for unicycle-type and two-steering-wheels mobile robots. *INRIA Report 2097*, 1993.

[12] P. Morin and C. Samson. Motion control of wheeled mobile robots. In B. Siciliano and O. Khatib, editors, *Springer Handbook of Robotics*, pp. 799-826. Springer, Berlin, Germany, 2008.

[13] S. Mukhopadhyay, C. Wang, M. Patterson, M. Malisoff, and F. Zhang. Collaborative autonomous surveys in marine environments affected by oil spills. In A. Koubaa and A. Khelil, editors, *Cooperative Robots and Sensor Networks 2014*, pp. 87-113. Springer, New York, 2014.

[14] F. Zhang, D. Fratantoni, D. Paley, J. Lund, and N. Leonard. Control of coordinated patterns for ocean sampling. *International Journal of Control*, 80(7):1186-1199, 2007.

[15] F. Zhang, E. Justh, and P. Krishnaprasad. Boundary following using gyroscopic control. In *Proc. IEEE Conf. Decision and Control*, Paradise Island, Bahamas, 2004, pp. 5204-5209.

[16] F. Zhang, A. O'Connor, D. Luebke, and P. Krishnaprasad. Experimental study of curvature-based control laws for obstacle avoidance. In *Proc. IEEE International Conf. Robotics and Automation*, New Orleans, LA, 2004, pp. 3849-3854.